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Quantum Gauge Theories*

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Abstract

The scope of this review is to give a pedagogical introduction to some new calculations and methods developed by the author in the context of quantum groups and their applications. The review is self-contained and serves as a "first aid kit" before one ventures into the beautiful but bewildering landscape of Woronowicz's theory. First, we present an up-to-date account of the methods and definitions used in quantum gauge theories. Then, we highlight our new results. The present paper is by no means an exhaustive overview of this swiftly developing subject.

Keywords: quantum groups, q-gauge theories, q-anti-de Sitter space, q-conformal correlation functions.

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Contents

1 Introduction

In the past two decades there has been an extremely rapid growth in the interest of quantum groups and their applications [1]. One can mention q-harmonic analysis and q-special functions [2], conformal field theories [3, 4, 5], in the vertex and spin models [6, 7], anyons [8, 9, 10], in quantum optics [11], in the loop approach of quantum gravity [12], in large N QCD [13] where the authors constructed the master field using $q = 0$ deformed commutation relations, in "fuzzy physics" [14] and quantum gauge theories [15, 16, 17, 18].

The quantum groups, found in the investigation of integrable systems, are a class of noncommutative noncocomutative Hopf algebras. They were studied by Faddeev and his collaborators [19]. The initial aim of these authors was to formulate a quantum theory of solitons [20]. Most of their definitions are inspired by the quantum inverse scattering method [21, 22, 23]. The term "quantum group" was introduced by Drinfel'd in [24]. It was considered as an invariance group by Sudbery [25]. A simple example of a noncommutative space is given by the Manin's plane [26]. In another direction Woronowicz [27], in his seminal paper, considered what he proposed to call pseudogroups[1] and studied bicovariant bimodules as objects analogue to tensor bundles over Lie groups. He has also introduced the theory of bicovariant differential calculus. This theory has turned out to be the appropriate language to study gauge theories based on noncommutative spaces.

Actually, there are maps [28, 29, 30, 31] relating the deformed gauge fields to the ordinary ones. These maps are the analogues of the Seiberg-Witten map [32]. We found these maps using the Gerstenhaber star product [33] instead of the Groenewold-Moyal star product [34]. In ref. [35], it was proposed that quantum fluctuations in the $\mathrm{AdS}_3\times\mathrm{S}^3$ background have the effect of deforming spacetime to a noncommutative manifold. The evidence is based on the quantum group interpretation of the cutoff on single particle chiral primaries. In Ref. [36], it was shown for the case of two-dimensional de Sitter space that there is a natural q-deformation of the conformal group, with q a root of unity, where the unitary principal series representations become finite-dimensional cyclic representations. In the framework of the dS/CFT correspondence, these representations can lead to a description with a finite

[1] A pseudogroup $G(\varrho_1, \varrho_2, .., \varrho_k)$ of a group G is a set with a binary operation which gradually acquires the group properties of G and gradually satisfies the group axioms as some of the parameters $\varrho_1, \varrho_2, .., \varrho_k$ of the set approach certain limiting values or tend asymptotically to infinity. In the case of a q-deformation the parameter is $\varrho = q$. In the limit $q \to 1$ the pseudogroup acquires the properties of a group.

dimensional Hilbert space and unitary evolution. The computation of the q-deformed metrics. of the q-deformed anti-de Sitter space AdS_5^q was carried out in [18]. The form of the q-deformed conformal correlation functions in four dimensions was explicitly found in [5]. Given these results, we hope to construct the q-deformed $\mathrm{AdS}_5/\mathrm{CFT}_4$ completely. Recently, Vafa and his collaborators [37], have counted the number of 4-dimensional BPS black holes states on local Calabi-Yau three-folds involving an arbitrary genus g Riemann surface and showed that the topological gauge theory on the brane reduces to a q-deformed $2d$ Yang-Mills theory. All these results prove that quantum groups have unexpected applications and will certainly shed light on still open questions in quantum gravity and quantum gauge theory.

This review is organized as follows. In section 2, the concepts of quantum groups are introduced. In section 3, we present the simple example of two dimensional q-deformed plane. In section 4, we recall the celebrated Woronowicz formalism. In section 5, we recall the $SO_q(6)$ bicovariant differential calculus. In section 6, we construct the quantum gauge transformations and the quantum BRST and anti-BRST transformations. Then, we introduce the quantum Batalin-Vilkovisky operator. In section 7, we introduce a map between q-deformed gauge fields and ordinary gauge fields. In section 8, we study the quantum anti-de Sitter space AdS_5^q. We compute the quantum metrics. and the linear transformations leading to them, both for q real and q a phase. In section 9, we compute the quantum conformal correlation functions. Section 10 is devoted to the conclusion.

2 Quantum Groups

Let us first consider a group G in the usual sense, i.e. a set satisfying the group axioms[2], and $\mathbb{C}$ be a field of complex numbers. With this group one can associate a commutative, associative $\mathbb{C}$-algebra of functions from G to $\mathbb{C}$ with pointwise algebra structure, i.e. for any two elements f and f', for any scalar $\alpha \in \mathbb{C}$, and $g \in G$ we have

$$\begin{aligned} (f+f')(g) &:= f(g) + f'(g), \\ (\alpha f)(g) &:= \alpha f(g), \\ (f f')(g) &:= f(g) f'(g). \end{aligned} \tag{1}$$

[2]If the invertibility condition is relaxed, we have only a semigroup.

If G is a topological group, usually only continuous functions are considered, and for an algebraic group[3] the functions are normally polynomials functions. These algebras are called "**algebras of functions on** G".
Example:
Let G be an arbitrary subgroup of the group $GL\left(N,\mathbb{C}\right)$. Let $Fun\left(G\right)$ be the algebra of complex valued functions on G. This algebra is unital with unit $1 : G \rightarrow \mathbb{C},\ g \rightarrow 1$ and is a $*$-algebra, where for all $f \in Fun\left(G\right)$ the function $f^*\left(g\right) = \overline{f\left(g\right)}$ for all $g \in G$. Consider now the matrix elements $M^a{}_b$ of the fundamental representation of G. For all a and b, the coefficient functions: $u^a{}_b : G \rightarrow \mathbb{C}, \quad g \rightarrow u^a{}_b\left(g\right) = M^a{}_b$ belong to $Fun\left(G\right)$.
The algebras (1) inherit some extra structures. Using the group structures of G, we can introduce on the set $\mathcal{A} = Fun(G)$ of complex-valued functions on G three other linear mappings, the coproduct Δ, the counit ϵ, the coinverse (or antipode) S:

$$\begin{aligned} \Delta f\left(g,g'\right) &= f\left(gg'\right), \quad \Delta : Fun\left(G\right) \rightarrow Fun\left(G\times G\right) \\ \epsilon\left(f\right) &= f\left(e\right), \qquad \epsilon : Fun\left(G\right) \rightarrow \mathbb{C}, \\ S\left(f\right)\left(g\right) &= g\left(g^{-1}\right), \quad S : Fun\left(G\right) \rightarrow Fun\left(G\right) \end{aligned} \tag{2}$$

where e is the unit of G.
In order to work with functions on G alone, we consider the tensor product $Fun\left(G\right) \otimes Fun\left(G\right)$ as a linear subspace of $Fun\left(G\times G\right)$ by identifying $f_1 \otimes f_2 \in Fun\left(G\right) \otimes Fun\left(G\right)$ with the functions $\left(f_1 \otimes f_2\right)\left(g,h\right) = f_1\left(g\right) f_2\left(h\right)$ on $G \times G$.
The linear mappings satisfy the relations:

$$\begin{aligned} \left(id \otimes \Delta\right) \circ \Delta &= \left(\Delta \otimes id\right) \circ \Delta \\ \left(id \otimes \epsilon\right) \circ \Delta &= \left(\epsilon \otimes id\right) \circ \Delta = id \\ m \circ \left(S \otimes id\right) \circ \Delta &= m\left(id \otimes S\right) \circ \Delta = \eta \circ \epsilon \end{aligned} \tag{3}$$

and

$$\begin{aligned} \Delta\left(ab\right) &= \Delta\left(a\right)\Delta\left(b\right), \quad \Delta\left(I\right) = I \otimes I \\ \epsilon\left(ab\right) &= \epsilon\left(a\right)\epsilon\left(b\right), \quad \epsilon\left(I\right) = 1 \\ S\left(ab\right) &= S\left(b\right)S\left(a\right), \quad S\left(I\right) = 1 \end{aligned} \tag{4}$$

where the linear mapping (unit) $\eta : \mathbb{C} \rightarrow \mathcal{A}$ is such that $\eta\left(1\right)$ is the unit I of $\mathcal{A}$, $a, b \in \mathcal{A}$ and $m : \mathcal{A} \rightarrow \mathcal{A}$ is the multiplication map $m\left(a \otimes b\right) = ab$. The

[3] A group G is called **algebraic** if it is provided with the structure of an algebraic variety in which the multiplication and the inversion mappings are regular mappings (morphisms) of algebraic varieties.

product in $\Delta(a)\Delta(b)$ is the product in $\mathcal{A}\otimes\mathcal{A}: (a\otimes b)(c\otimes d) = ac\otimes bd$.
The relations (3) and (4) define the **Hopf algebra structures** [39].
For the coordinates functions $u^a_b : G \to \mathbb{C}$ we have

$$\Delta u^a_b(g,h) = u^a_b(gh) = (gh)^a_b = \sum_c g^a_c h^c_b = u^a_c(g)\, u^c_b(h)\,. \tag{5}$$

In general a coproduct can be expanded on $\mathcal{A}\otimes\mathcal{A}$ as:

$$\Delta(a) = \sum_i a^i_1 \otimes a^i_2, \qquad a^i_1, a^i_2 \in \mathcal{A}. \tag{6}$$

Using Sweedler notation [40], we shall suppress the index i and write this sum as:

$$\Delta(a) = \sum a_{(1)} \otimes a_{(2)}. \tag{7}$$

Here the subscripts (1), (2) refer to the corresponding tensor factors.
Now the algebra is **deformed** or **quantized**, i.e. the algebra structure is changed so that the algebra is not commutative anymore, but the extra structures and axioms remain the same. This algebra is called " **algebras of functions on a quantum group**", Another definition of quantum groups is given by:

Definition:
A quantum group is a **quasitriangular** Hopf algebra. This a pair $(\mathcal{A}, R)$ where $\mathcal{A}$ is a Hopf algebra and R is an invertible element of $\mathcal{A}\otimes\mathcal{A}$ such that $(\Delta\otimes id)(R) = R^{13}R^{23}$ and $\Delta'(a) = R\Delta(a)R^{-1}$ for $a\in\mathcal{A}$.
Here Δ' is the opposite comultiplication and R^{12}, R^{13}, R^{23} are defined as follows: If $R = \sum_i x_i\otimes y_i$, where $x_i, y_i \in \mathcal{A}$ then $R^{12} = \sum_i x_i \otimes y_i \otimes 1$, $R^{13} = \sum_i x_i \otimes 1 \otimes y_i$, $R^{23} = \sum_i 1\otimes x_i \otimes y_i$.

There are three ways of considering algebras of functions on a group and their deformations:

(a) polynomial functions $Poly(G)$ (developed by Woronowicz and Drinfel'd)

(b) continuous functions $C(G)$, if G is a topological group (developed by Woronowicz)

(c) formal power series (developed by Drinfel'd).

There is a similar concept of "**quantum spaces**": If G acts on a set X (e.g. a vector space), there is a corresponding so-called **coaction** of the commutative algebra of functions on G on the commutative algebra of functions on X satisfying certain axioms. The latter algebra can often be deformed into a non-commutative algebra called the " **algebra of functions on a**

quantum space".

If we consider a compact Hausdorff space X, and the set $C(X)$ of continuous, complex valued functions on X. $C(X)$ is naturally endowed with the structure of a commutative algebra with unit over the complex number field, equipped moreover with anti-linear involution $*$ given by

$$(f^*)(p) = \overline{f(p)} \tag{8}$$

and the norm

$$\|f\| = \sup_{p \in X} |f(p)| . \tag{9}$$

This norm can be seen to obey the condition

$$\|f^* f\| = \|f\|^2 . \tag{10}$$

Algebras with the above properties are known as C^*-algebras. We see therefore that every compact Hausdorff space X is in a natural way associated with a commutative C^*-algebra with unit[4], namely $C(X)$. Moreover, every continuous mapping between compact Hausdorff spaces, $T : X \to Y$, determines a C^*-homomorphism: $T^* : C(Y) \to C(X)$, given by

$$(T^* f)(p) = f(T(p)) . \tag{11}$$

Points of Y correspond to linear multiplicative functionals on $C(Y)$. The **Gelfand-Naimark** theorem [41] states that this correspondence is one to one[5]. Consequently, it is natural to make the following generalization: A compact quantum space corresponds, by an extension of this isomorphism, to a noncommutative $C^\star$-algebra with unit.

3 Manin's Construction

A simple example of a quantum space is given by the Manin's plane[6]. The quantum plane $R_q[2,0]$ is defined, according to Manin [26], in terms of two

[4]Every commutative $C^\star$-algebra may be identified with an algebra of continuous functions on a locally compact topological space. If the algebra has a unit element, this space is moreover compact. In the contrary case, we are dealing with the algebra of continuous functions on noncompact space, subject to the condition of vanishing at infinity.

[5]In the language of category theory: there exists a contravariant isomorphism between the category of compact topological spaces and that of $C^\star$ -algebras with unit.

[6]The quantum plane approach was first suggested by Yu Kobyzev (Moscow, winter 1986 and developed by Manin at université de Montréal in June 1988 .

variables $\hat{x}$, $\hat{y}$, which satisfy the commutation relations

$$\hat{x}\hat{y} = q\hat{y}\hat{x} \tag{12}$$

where q is a complex number[7]. The coordinates neither commute nor anticommute unless $q = \pm 1$, respectively. Now consider a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_q(2) \tag{13}$$

such that

$$\begin{aligned} \hat{x}' &= a\hat{x} + b\hat{y} \\ \hat{y}' &= c\hat{x} + d\hat{y} \end{aligned} \tag{14}$$

and $(\hat{x}', \hat{y}') \in R_q[2,0]$. The elements of M are supposed to commute with x, y. This condition imposes restrictions upon M, giving the $GL_q(2)$ relations

$$\begin{aligned} ab &= qba, \qquad cd = qdc, \\ ac &= qca, \qquad bc = cb, \\ bd &= qdb, \qquad ad - da = \left(q - q^{-1}\right) bc. \end{aligned} \tag{15}$$

The classical case is obtained by setting q equal to one.

Using these relations, it is easy to show that $D_q = ad - qbc$ commutes with all the elements a, b, c, d and thus may be considered as a number, the "quantum determinant[8]". The choice $D_q = 1$ restricts the quantum group to $SL_q(2)$. Because D_q commutes with elements of M there exists an inverse

$$M^{-1} = (D_q)^{-1} \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}, \tag{16}$$

which is both a left and right inverse for M. Note that M^{-1} is a member of $GL_{q^{-1}}(2)$ rather than $GL_q(2)$, and thus $GL_q(2)$ is not, strictly speaking, a

[7] Manin uses q^{-1} where we use q, following the usage of the Leningrad school [19].

[8] The quantum determinant is defined as the determinant of the middle matrix in the Borel decomposition:
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix}$.

group. The algebra (15) is associative under multiplication and the relations may be reexpressed in a tensor form

$$R^{ij}{}_{kl}M^{k}{}_{m}M^{l}{}_{n}=M^{i}{}_{k}M^{j}{}_{l}R^{kl}{}_{mn} \tag{17}$$

where $R^{ij}{}_{kl}=R^{ij}{}_{kl}\left(q\right)$ is a braiding matrix, whose explicit form is given by[9]

$$R^{ij}{}_{kl}=\left(\begin{array}{cccc}1&0&0&0\\0&0&q&0\\0&q&1-q^{2}&0\\0&0&0&1\end{array}\right). \tag{18}$$

Let us give the example of $U_{q}\left(2\right)$ obtained by requiring that the unitary condition hold for this 2×2 quantum matrix:

$$M^{n\dagger}{}_{m}=M^{n}{}_{m}^{-1}. \tag{19}$$

The 2×2 matrix belonging to $U_{q}\left(2\right)$ preserves the nondegenerate bilinear form [42] C_{nm}

$$C_{nm}M^{n}{}_{k}M^{m}{}_{l}=D_{q}C_{kl},\quad C^{nm}M^{k}{}_{n}M^{l}{}_{m}=D_{q}C^{kl},\quad C_{kn}C^{nl}=\delta^{l}_{k}, \tag{20}$$

$$C_{nm}=\left(\begin{array}{cc}0&-q^{-1/2}\\q^{1/2}&0\end{array}\right),\quad C^{nm}=\left(\begin{array}{cc}0&q^{-1/2}\\-q^{1/2}&0\end{array}\right). \tag{21}$$

The algebra $Fun\left(U_{q}\left(2\right)\right)$ is freely generated by the associative unital C^{*}-algebra. $Fun\left(U_{q}\left(2\right)\right)$ is a Hopf algebra with comultiplication Δ, counit ϵ and antipode S which are given by:

---

[9] Castellani [1] and other authors use the following relation

$\mathcal{R}^{ij}{}_{kl}\left(q\right)M^{k}{}_{m}M^{l}{}_{n}=M^{i}{}_{k}M^{j}{}_{l}\mathcal{R}^{lk}{}_{mn}\left(q\right)$ where $\mathcal{R}^{ij}{}_{kl}\left(q\right)=\left(\begin{array}{cccc}q&0&0&0\\0&1&0&0\\0&q-q^{-1}&1&0\\0&0&0&q\end{array}\right).$

This matrix appeared for the first time in the paper [43]. The relation between the two matrices is given by $\mathcal{R}^{ij}{}_{kl}\left(q\right)=qR^{ij}{}_{lk}\left(q^{-1}\right)=q\left(R^{-1}\right)^{ij}_{kl}\left(q\right)$.

The matrix $\mathcal{R}^{ij}{}_{kl}$ satisfies the well known Yang-Baxter relation [44, 45]:
$\mathcal{R}^{a_1b_1}{}_{a_2b_2}\mathcal{R}^{a_2c_1}{}_{a_3c_2}\mathcal{R}^{b_2c_2}{}_{b_3c_3}=\mathcal{R}^{b_1c_1}{}_{b_2c_2}\mathcal{R}^{a_1c_2}{}_{a_2c_3}\mathcal{R}^{a_2b_2}{}_{a_3b_3}$.

If $(\mathcal{A},\mathcal{R})$ is a quasitriangular Hopf algebra, then $\mathcal{R}$ satisfies the Yang-Baxter equation.

The Yang Baxter equation and the noncommutativity of the elements $M^{n}{}_{m}$ lie at the foundation of the method of commuting transfer-matrices in classical statistical mechanics [45] and factorizable scattering theory [44, 46], the quantum theory of magnets [47] and the inverse scattering method for solving nonlinear equations of evolutions [48].

-comultiplication (also called coproduct)

$$\Delta\left(M^n{}_m\right) = M^n{}_k \otimes M^k{}_m. \tag{22}$$

This coproduct Δ on $Fun\left(U_q\left(2\right)\right)$ is directly related, for $q = 1$ (the nondeformed case), to the pullback induced by left multiplication of the group on itself.

-co-unit ϵ

$$\varepsilon\left(M^n{}_m\right) = \delta^n_m \tag{23}$$

-antipode S (coinverse)

$$S\left(M^n{}_k\right) M^k{}_m = M^n{}_k S\left(M^k{}_m\right) = \delta^n_m \tag{24}$$

$$S\left(M^n{}_m\right) = \frac{1}{D_q} C^{nk} M^l{}_k C_{lm}. \tag{25}$$

With the nondegenerate form C the R matrix has the form

$$R^{+nm}{}_{kl} = R^{nm}{}_{kl} = \delta^n_k \delta^m_l + q C^{nm} C_{kl}, \tag{26}$$

$$R^{-nm}{}_{kl} = R^{-1nm}{}_{kl} = \delta^n_k \delta^m_l + q^{-1} C^{nm} C_{kl}. \tag{27}$$

The R matrices satisfy the Hecke relations

$$R^{\pm 2} = \left(1 - q^{\pm 2}\right) R^{\pm} + q^{\pm 2}\mathbf{1} \tag{28}$$

and the relations

$$C_{nm} R^{\pm an}{}_{kc} R^{\pm cm}{}_{lb} = q^{\pm 1} \delta^a_b C_{kl}. \tag{29}$$

4 Review of Woronowicz's Bicovariant Differential Calculus

In this section we give a short review of the bicovariant differential calculus on quantum groups as developed by Woronowicz [27].

Definition:
A **first-order differential calculus over an algebra** $\mathcal{A}$ is a pair (Γ, d) such that:

(1) Γ is an $\mathcal{A}$-bimodule, i.e. $(a\omega)\, b = a\, (\omega b)$

for all $a, b \in \mathcal{A}$, $\omega \in \Gamma$, where the left and right actions which make Γ, respectively a left $\mathcal{A}$-module and a right $\mathcal{A}$-module are written multiplicatively;

(2) d is a linear map, $d : \mathcal{A} \to \Gamma$;

(3) for any $a, b \in \mathcal{A}$, the Leibniz rule is satisfied, i.e.

$$d\,(ab) = d\,(a)\, b + ad\,(b) \tag{30}$$

(4) the bimodule Γ, or "space of one-forms", is spanned by elements of the form adb, $a, b \in \mathcal{A}$.

Definition:
A **bicovariant bimodule** over a Hopf algebra $\mathcal{A}$ is a triple $(\Gamma, \Delta_L, \Delta_R)$ such that :

(1) Γ is an $\mathcal{A}$-bimodule;

(2) Γ is an $\mathcal{A}$-bicomodule with left and right coactions Δ_L and Δ_R respectively, i.e.

$$(id \otimes \Delta_L) \circ \Delta_L = (\Delta \otimes id) \circ \Delta_L \qquad (\epsilon \otimes id) \circ \Delta_L = id \tag{31}$$

making Γ a left $\mathcal{A}$-comodule,

$$(\Delta_R \otimes id) \circ \Delta_R = (id \otimes \Delta) \circ \Delta_R \qquad (id \otimes \epsilon) \circ \Delta_R = id \tag{32}$$

making Γ a right $\mathcal{A}$-comodule, and

$$(id \otimes \Delta_R) \circ \Delta_L = (\Delta_L \otimes id) \circ \Delta_R \tag{33}$$

which is a the $\mathcal{A}$-bicomodule property;

(3) the coactions [10], Δ_L and Δ_R are bimodule maps, i.e.

[10] The left (resp. right) coactions are pullbacks for one forms induced by left (resp. right) multplication of the group by itself.

$$
\begin{aligned}
\Delta_L(a\omega b) &= \Delta(a)\,\Delta_L(\omega)\,\Delta(b)\\
\Delta_R(a\omega b) &= \Delta(a)\,\Delta_R(\omega)\,\Delta(b)
\end{aligned} \tag{34}
$$

Remark:
The Sweedler notation for coproducts in the Hopf algebra $\mathcal{A}$ is taken to be $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for all $a \in \mathcal{A}$ and is extended to the coactions as $\Delta_L(\omega) = \omega_{(\ \mathcal{A})} \otimes \omega_{(\Gamma)}$ and $\Delta_R(\omega) = \omega_{(\Gamma)} \otimes \omega_{(\mathcal{A})}$.

Definition:
A **first-order bicovariant differential calculus over a Hopf algebra** $\mathcal{A}$ is a quadruple $(\Gamma, d, \Delta_L, \Delta_R)$ such that:

(1) (Γ, d) is a first-order differential calculus over $\mathcal{A}$;

(2) $(\Gamma, \Delta_L, \Delta_R)$ is a bicovariant bimodule over $\mathcal{A}$;

(3) d is both a left and a right comodule map, i.e.

$$
\begin{aligned}
(id \otimes d) \circ \Delta(a) &= \Delta_L(da)\\
(d \otimes id) \circ \Delta(a) &= \Delta_R(da)
\end{aligned} \tag{35}
$$

for all $a \in \mathcal{A}$.

5 $SO_q(6)$ Bicovariant Differential Calculus

In this section, we construct the left invariant vector fields, the quantum trace and the quantum Killing metric which are the important ingredients in the construction of quantum gauge theories.
Let us first consider the bicovariant bimodule Γ over the quantum group $SO_q(6)$. This quantum group is the symmetry group of the q-deformed AdS/CFT correspondence (to be constructed). The corresponding braiding matrix is given by:

$$
\begin{aligned}
\hat{R} &= q\sum_{\substack{i=-3\\ i\neq 0}}^{3} \delta_i^i \otimes \delta_i^i + \sum_{\substack{i,j=-3\\ i\neq j,-j}}^{3} \delta_i^i \otimes \delta_j^j + q^{-1}\sum_{\substack{i=-3\\ i\neq 0}}^{3} \delta_{-i}^{-i} \otimes \delta_i^i\\
&+ k\sum_{\substack{i,j=-3\\ i>j}}^{3} \delta_j^i \otimes \delta_i^j - k\sum_{\substack{i,j=-3\\ i>j}}^{3} q^{\rho_i-\rho_j}\delta_j^i \otimes \delta_{-j}^{-i}
\end{aligned} \tag{36}
$$

where we have used the notations:

$$
\begin{aligned}
k &\equiv q - q^{-1} \\
\rho_i &= (2, 1, 0, 0, -1, -2).
\end{aligned} \tag{37}
$$

The matrix elements $\hat{R}$ vanish unless the indices satisfy the following conditions:

$$
\begin{array}{lllll}
\text{either} & i \neq -j & \text{and} & k = i,\ l = j, & \text{or} \quad l = i, k = j \\
\text{or} & i = -j & \text{and} & k = -l. &
\end{array} \tag{38}
$$

The matrix $\hat{R}$ enters in local representations of the Birman-Wenzel-Murakami algebra [49]. $\hat{R}$ admits a projector decomposition [19]:

$$
\hat{R} = qP_S - q^{-1}P_a + q^{-5}P_t, \tag{39}
$$

where P_S, P_a, P_t are the projections operators onto three eigenspaces of $\hat{R}$ with dimensions respectively 20, 15, 1: they project the tensor product $x \otimes x$ of the fundamental corepresentation x of $SO_q(6)$ into the corresponding irreducible corepresentations:

$$
\begin{aligned}
P_S &= \frac{1}{q + q^{-1}} \left[\hat{R} + q^{-1} I - \left(q^{-1} + q^{-5} \right) P_t \right], \\
P_a &= \frac{1}{q + q^{-1}} \left[-\hat{R} + qI - \left(q + q^{-5} \right) P_t \right], \\
(P_t)^{ab}_{cd} &= \left(C_{ef} C^{ef} \right)^{-1} C_{cd} C^{ab}.
\end{aligned} \tag{40}
$$

The $\hat{R}$ matrices satisfy the Yang-Baxter equation.

$$
\hat{R}^{ij}{}_{pq} \hat{R}^{qk}{}_{mn} \hat{R}^{pm}{}_{rl} = \hat{R}^{jk}{}_{mp} \hat{R}^{im}{}_{rq} \hat{R}^{qp}{}_{ln} \tag{41}
$$

and the relations:

$$
\begin{aligned}
C_{qm} \hat{R}^{\pm qp}{}_{ji} \hat{R}^{\pm ml}{}_{pk} &= \delta^l_j C_{ik} \\
C^{qm} \hat{R}^{\pm ip}{}_{jq} \hat{R}^{\pm kl}{}_{pm} &= \delta^l_j C^{ik}.
\end{aligned} \tag{42}
$$

The noncommutativity of the elements $M^i{}_j$ is expressed as

$$
\hat{R}^{ij}{}_{pq} M^p{}_l M^q{}_n = M^i{}_q M^j{}_p \hat{R}^{qp}{}_{ln}. \tag{43}
$$

The generators $M^i{}_j$ satisfy the orthogonality condition

$$
C_{ij} M^i{}_q M^j{}_p = C_{qp}. \tag{44}
$$

The nondegenerate bilinear form C^{nm} is given by

$$C^{nm}=\left(\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & q^{-2} \\ 0 & 0 & 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 \\ q^{2} & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{45}$$

Now, let us consider the fundamental bimodule over $Fun\left(SO_q\left(6\right)\right)$ generated by left invariant basis θ^a, $a=1,...,6$. The right coaction is defined as

$$\Delta_R\left(\theta^a\right)=\theta^b\otimes S\left(M^a_{\ b}\right). \tag{46}$$

We can also define the conjugate $\left(\theta^{*a}\right)=\left(\theta^a\right)^*\equiv\bar{\theta}_a$. The right coaction acts on this basis as

$$\Delta_R\left(\bar{\theta}_a\right)=\bar{\theta}_b\otimes M^b_{\ a}. \tag{47}$$

This equation is easily obtained from Equ. (46) by the antilinear $*$ involution using the relation $\left(\Delta_R\left(\theta^a\right)\right)^*=\Delta_R\left(\theta^a\right)^*, \quad \left(M^a_{\ b}\right)^*\equiv M^{\dagger b}_{\ \ a}=S\left(M^b_{\ a}\right)$. There exist linear functionals $f^a_{\ b}$ and $\bar{f}^a_{\ b}:Fun\left(SO_q\left(6\right)\right)\rightarrow\mathbb{C}$ for the left basis θ^a and $\bar{\theta}_a$ such that

$$\begin{aligned} \theta^a M^n_{\ m} &= \left(f^a_{\ b}\star M^n_{\ m}\right)\theta^b=\left(id\otimes f^a_{\ b}\right)\Delta\left(M^n_{\ m}\right)\theta^b \\ &= f^a_{\ b}\left(M^k_{\ m}\right)M^n_{\ k}\theta^b, \end{aligned} \tag{48}$$

$$\begin{aligned} M^n_{\ m}\theta^a &= \theta^b\left(f^a_{\ b}\circ S\star M^n_{\ m}\right)=\theta^b\left(id\otimes f^a_{\ b}\circ S\right)\Delta\left(M^n_{\ m}\right) \\ &= f^a_{\ b}\left(S\left(M^k_{\ m}\right)\right)\theta^b M^n_{\ k}, \end{aligned} \tag{49}$$

$$\begin{aligned} \bar{\theta}_a M^n_{\ m} &= \left(\bar{f}^b_{\ a}\star M^n_{\ m}\right)\bar{\theta}_b=\left(id\otimes\bar{f}^b_{\ a}\right)\Delta\left(M^n_{\ m}\right)\bar{\theta}_b \\ &= \bar{f}^b_{\ a}\left(M^k_{\ m}\right)M^n_{\ k}\bar{\theta}_b, \end{aligned} \tag{50}$$

$$\begin{aligned} M^n_{\ m}\bar{\theta}_a &= \bar{\theta}_b\left(\bar{f}^b_{\ a}\circ S\star M^n_{\ m}\right)=\bar{\theta}_b\left(id\otimes\bar{f}^b_{\ a}\circ S\right)\Delta\left(M^n_{\ m}\right) \\ &= \bar{f}^b_{\ a}\left(S\left(M^k_{\ m}\right)\right)\bar{\theta}_b M^n_{\ k}. \end{aligned} \tag{51}$$

The orthogonality condition Equ. (44) must be consistent with the bimodule structure. This implies that

$$f^a_{\ b}\left(C_{ij}M^i_{\ q}M^j_{\ p}\right)=C_{ij}f^a_{\ c}\left(M^i_{\ q}\right)f^c_{\ b}\left(M^j_{\ p}\right)=\delta^a_b C_{qp}. \tag{52}$$

Comparing with Equ. (42) we get two solutions

$$
\begin{aligned}
f_{+\ b}^{a}\left(M^{n}{}_{m}\right) &= \hat{R}^{an}{}_{mb} &(53)\\
f_{-\ b}^{a}\left(M^{n}{}_{m}\right) &= \hat{R}^{-an}{}_{mb}. &(54)
\end{aligned}
$$

Applying the $*$-operation on both sides of Equ. (48) and substituting $\left(M^{a}{}_{b}\right)^{*} \equiv M^{\dagger b}{}_{a} = S\left(M^{b}{}_{a}\right)$ we get

$$
\bar{f}^{a}{}_{\pm b}\left(S\left(M^{n}{}_{m}\right)\right) = f^{a}{}_{\mp b}\left(M^{n}{}_{m}\right) = \hat{R}^{\mp an}{}_{mb}. \tag{55}
$$

The representation with upper index of $\bar{\theta}_{a}$ is defined by the bilinear form C:

$$
\bar{\theta}_{\pm}^{b} = \bar{\theta}_{\pm a} C^{ab}. \tag{56}
$$

Then the right coaction is

$$
\Delta_{R}\left(\bar{\theta}_{\pm}^{b}\right) = \bar{\theta}_{\pm}^{a} \otimes C_{ad} M^{d}{}_{e} C^{eb} = \bar{\theta}_{\pm}^{a} \otimes S^{-1}\left(M^{b}{}_{a}\right), \tag{57}
$$

where the inverse of the antipode S^{-1} satisfies

$$
S^{-1} S\left(M^{a}{}_{b}\right) = M^{a}{}_{b}. \tag{58}
$$

The functionals $\tilde{f}^{a}{}_{\pm b}$ corresponding to the basis $\bar{\theta}_{\pm}^{b}$ are given by

$$
\tilde{f}^{a}{}_{\pm b} = C_{bd} \bar{f}^{d}{}_{e} C^{ea}. \tag{59}
$$

The transformation of the adjoint representation for the quantum group acts on the generators $M^{a}{}_{b}$ as the right adjoint coaction[11] Ad_{R} :

$$
Ad_{R}\left(M^{n}{}_{m}\right) = M^{l}{}_{k} \otimes S\left(M^{n}{}_{l}\right) M^{k}{}_{m}. \tag{60}
$$

A bicovariant bimodule which includes the adjoint transformation Γ_{Ad} is obtained by taking the tensor product $\theta^{n} \otimes \bar{\theta}_{m} \equiv \theta^{n}{}_{m}$ of two fundamental modules.
The right coaction Δ_{R} on the basis $\theta^{n}{}_{m}$ is

$$
\Delta_{R}\left(\theta^{n}{}_{m}\right) = \theta^{l}{}_{k} \otimes S\left(M^{n}{}_{l}\right) M^{k}{}_{m}. \tag{61}
$$

[11]Example: Let $\mathcal{A} = \mathcal{O}(G)$ a coordinate Hopf algebra, then

$$
\begin{aligned}
\forall g, h \in G, Ad_{R}\left(M^{i}{}_{j}\right)(g,h) &= \sum M^{k}{}_{l}(g)\left(S\left(M^{i}{}_{k}\right) M^{l}{}_{j}\right)(h) = \sum g^{k}{}_{l}\left(h^{-1}\right)^{i}{}_{k} h^{l}{}_{j}\\
&= \left(h^{-1} g h\right)^{i}{}_{j} = M^{i}{}_{j}\left(h^{-1} g h\right).
\end{aligned}
$$

With the requirement that the $*$-operation be a module antiautomorphism $(\Gamma_{Ad})^* = \Gamma_{Ad}$ we can find two different types of left invariant basis containing the adjoint representation. We take $\theta^n_{+\ \ m} = \theta^n_+ \bar{\theta}_{+m}$.

We introduce the left invariant basis θ^{ab} with upper indices:

$$\theta^{ab} = \theta^a{}_c C^{cb}. \tag{62}$$

In this basis the right coaction is given by

$$\Delta_R\left(\theta^{ab}\right) = \theta^{cd} \otimes S\left(M^a{}_c\right) S^{-1}\left(M^b{}_d\right). \tag{63}$$

The relation between the left and right multiplication for this basis is

$$\begin{aligned}
\theta^{ab} M^n{}_m &= \left(\theta^a \otimes \bar{\theta}^b\right) M^n{}_m = \theta^a \otimes \left(\tilde{f}^b{}_d \star M^n{}_m\right) \bar{\theta}^d \\
&= \left(f^a{}_c \star \left(\tilde{f}^b{}_d \star M^n{}_m\right)\right) \left(\theta^c \otimes \bar{\theta}^d\right) \\
&= \left(f^{ab}_{Ad\ \ cd} \star M^n{}_m\right) \theta^{cd} = f^{ab}_{Ad\ \ cd}\left(M^k{}_m\right) M^n{}_k \theta^{cd}
\end{aligned} \tag{64}$$

where

$$\begin{aligned}
f^{ab}_{Ad\ \ cd}\left(M^n{}_m\right) &= \left(f^a{}_c \star \tilde{f}^b{}_d\right)\left(M^n{}_m\right) = \left(f^a{}_c \otimes \tilde{f}^b{}_d\right) \Delta\left(M^n{}_m\right) \\
&= f^a{}_c\left(M^n{}_k\right) \tilde{f}^b{}_d\left(M^k{}_m\right) = \hat{R}^{an}{}_{kc} C_{de} \bar{f}^e{}_f \left(M^k{}_m\right) C^{fb} \\
&= \hat{R}^{an}{}_{kc} \hat{R}^{-bk}{}_{md}.
\end{aligned} \tag{65}$$

The exterior derivative d is defined as

$$\begin{aligned}
dM^n{}_m &= \frac{1}{\mathcal{N}}\left[X, M^n{}_m\right]_- = \left(\chi_{ab} \star M^n{}_m\right) \theta^{ab} \\
&= \left(id \otimes \chi_{ab}\right) \Delta\left(M^n{}_m\right) \theta^{ab} = \chi_{ab}\left(M^k{}_m\right) M^n{}_k \theta^{ab}
\end{aligned} \tag{66}$$

where $X = C_{ab}\theta^{ab}$ is the singlet representation of θ^{ab} and is both left and right co-invariant, $\mathcal{N} \in \mathbb{C}$ is a normalization constant which we take purely imaginary $\mathcal{N}^* = -\mathcal{N}$ and χ_{ab} are the quantum analogue of left-invariant vector fields given by:

$$\begin{aligned}
\chi_{ab}\left(M^n{}_m\right) &= \frac{1}{\mathcal{N}}\left(C_{cd} f^{cd}_{Ad\ ab}\left(M^n{}_m\right) - \delta^n_m C_{ab}\right) \\
&= \frac{1}{\mathcal{N}}\left(C_{cd} \hat{R}^{cn}{}_{ka} \hat{R}^{-dk}{}_{mb} - \delta^n_m C_{ab}\right).
\end{aligned} \tag{67}$$

Higher order differential calculus is built from the first order differential calculus by using the tensor product $\Gamma_{Ad} \otimes \Gamma_{Ad} \otimes ... \otimes \Gamma_{Ad}$. The basic operation is the bicovariant bimodule automorphism $\Lambda : \Gamma^{\otimes 2} \to \Gamma^{\otimes 2}$, defined as

$$\Lambda\left(\theta^{ab} \otimes \omega^{cd}\right) = \omega^{cd} \otimes \theta^{ab}, \tag{68}$$

where ω^{ab} is the right invariant basis defined by

$$\omega^{ab} = M^a{}_c M^b{}_d \theta^{cd}. \tag{69}$$

Let I represents a set of indices $I = (a, b)$, we write Equ. (69) as

$$\omega^I = T^I{}_J \theta^J, \tag{70}$$

where $T^I{}_J = T^{ab}{}_{cd} = M^a{}_c M^b{}_d$.
Using Equ. (69) and the definition of a bicovariant bimodule automorphism i.e. $\Lambda\left(a\tau b\right) = a\Lambda\left(\tau\right)b, \quad \forall a, b \in \Gamma_{Ad}^{\otimes 2}$:

$$\begin{aligned} \Lambda\left(\theta^I \otimes T^J{}_K \theta^K\right) &= T^J{}_K \theta^K \otimes \theta^I \\ &= f_{Ad}^I{}_N \left(T^L{}_K\right) T^J{}_L \Lambda\left(\theta^N \otimes \theta^K\right), \end{aligned} \tag{71}$$

which gives

$$\begin{aligned} \Lambda\left(\theta^M \otimes \theta^L\right) &= f_{Ad}^M{}_J \left(S\left(T^L{}_K\right)\right)\left(\theta^K \otimes \theta^J\right) \\ &= \Lambda^{ML}{}_{KJ}\left(\theta^K \otimes \theta^J\right). \end{aligned} \tag{72}$$

Therefore, the matrix representation of Λ on the basis $\theta^{ab} \otimes \theta^{cd}$ is

$$\Lambda^{ML}{}_{KJ} = f_{Ad}^M{}_J \left(S\left(T^L{}_K\right)\right), \tag{73}$$

leading to

$$\Lambda^{ijkl}{}_{ghef} = f_{Ad}^{ij}{}_{ef}\left(S\left(M^k{}_g M^l{}_h\right)\right). \tag{74}$$

The action of the exterior derivative d on $\mathcal{A}$ can be generalized on p-forms as in the usual differential calculus:

$$d : \Gamma_{Ad}^{\wedge p} \to \Gamma^{\wedge p+1}, \tag{75}$$

and is defined by $\forall \Omega \in \Gamma_{Ad}^{\wedge p}$:

$$d\Omega \equiv \frac{1}{\mathcal{N}}\left[X, \Omega\right]_{\pm} = \frac{1}{\mathcal{N}}\left(X \wedge \Omega - (-1)^p \Omega \wedge X\right). \tag{76}$$

The external product is given by

$$\theta^{ab} \wedge \theta^{cd} = \left(\delta_e^a \delta_f^b \delta_g^c \delta_h^d - \Lambda^{abcd}{}_{efgh}\right)\left(\theta^{ef} \otimes \theta^{gh}\right). \tag{77}$$

The two-form has been defined as

$$\Gamma_{Ad}^{\wedge 2} = \Gamma_{Ad}^{\otimes 2} / \left[\mathrm{Ker}\left(\Lambda - 1\right) \oplus \mathrm{Ker}\left(\Lambda - q^{-6} I\right)\right]. \tag{78}$$

This equation can be expressed in terms of the projectors and gives the Cartan-Maurer equations.
The quantum commutators of the quantum Lie algebra generators χ_{ab} are defined as

$$[\chi_{ab}, \chi_{cd}] \left(M^i{}_j\right) = (1-\Lambda)^{efgh}{}_{abcd} \left(\chi_{ef} \star \chi_{gh}\right) \tag{79}$$

and can be written as

$$\begin{aligned} [\chi_{ab}, \chi_{cd}] \left(M^i{}_j\right) &= \left(\chi_{ab} \otimes \chi_{cd}\right) Ad_R \left(M^i{}_j\right) \\ &= \chi_{ab} \left(M^l{}_n\right) \otimes \chi_{cd} \left(S\left(M^i{}_l\right) M^n{}_j\right) \\ &= C_{abcd}{}^{ef} \chi_{ef} \left(M^i{}_j\right), \end{aligned} \tag{80}$$

where $C_{abcd}{}^{ef}$ are the quantum structure constants.

To construct a quantum gauge invariant Lagrangian, we need a well defined quantum trace. We require that this trace is invariant under the right adjoint coaction:

$$Tr\left(M^i{}_j\right) = Tr\left(Ad_R\left(M^i{}_j\right)\right) = Tr\left(M^l{}_n \otimes S\left(M^i{}_l\right) M^n{}_j\right). \tag{81}$$

This equation is fulfilled if one defines the quantum trace as

$$Tr\left(M^i{}_j\right) = -C_{nk} M^n{}_m C^{mk}. \tag{82}$$

The quantum trace allows us to introduce the quantum Killing metric as in the usual undeformed case ($q = 1$)

$$g_{abcd} = Tr\left(\chi_{ab}\left(M^n{}_k\right) \chi_{cd}\left(M^k{}_m\right)\right). \tag{83}$$

Before we close this section, we note that the same construction can be done using the adjoint representation $M_b{}^a$ of the quantum group (see the paper of Aschieri and Castellani [1]). This allows us to find the relation between the Cartan-Maurer forms $\tilde{\theta}^i{}_j$ and the forms obtained by taking the tensor product $\theta^{ab} = \theta^a \bar{\theta}^b$.

Let $\tilde{\theta^a}$ be a left invariant basis of ${}_{\rm inv}\Gamma$, the linear subspace of all left-invariant elements of Γ i.e. $\Delta_L\left(\tilde{\theta^a}\right) = I \otimes \tilde{\theta^a}$. In the case $q = 1$ the left coaction Δ_L coincides with the pullback for 1-forms induced by the left multiplication of the group on itself. There exists an adjoint representation $M_b{}^a$ of the quantum group, defined by the right coaction Δ_R on the left-invariant $\tilde{\theta^a}$:

$$\Delta_R\left(\tilde{\theta^a}\right) = \tilde{\theta^b} \otimes M_b{}^a, \qquad M_b{}^a \in \mathcal{A} \tag{84}$$

where $\mathcal{A}$ is an associative unital $\mathbb{C}$-algebra. In the classical case, $M_b{}^a$ is indeed the adjoint representation of the group $SO(6)$. We recall that in this limit the left-invariant 1-forms $\tilde{\theta^a}$ can be constructed as

$$\tilde{\theta^a}(y)\,T_a = \left(y^{-1}dy\right)^a \qquad y \in SO(6)\,. \tag{85}$$

Under the right multiplication by a (constant) element $x \in SO(6) : y \to x$ we have[12],

$$\begin{aligned}\tilde{\theta^a}(yx)\,T_a &= \left[x^{-1}y^{-1}d(yx)\right]^a T_a = \left[x^{-1}\left(y^{-1}dy\right)x\right]^a T_a \\ &= \left[x^{-1}T_b x\right]^a \left(y^{-1}dy\right)^b T_a = M_b{}^a(x)\,\tilde{\theta^b}(y)\,T_a,\end{aligned} \tag{86}$$

so that

$$\tilde{\theta^a}(yx) = \tilde{\theta^b}(y)\,M_b{}^a(x) \tag{87}$$

or

$$R^\star\tilde{\theta^a}(y,x) = \tilde{\theta^b}M_b{}^a(y,x)\,, \tag{88}$$

which reproduces Equ.(84) for $q = 1$ ($R^\star$ is the analogue of the right coaction Δ_R in this limit and is defined via the pullback $R_x^\star$ on functions or 1-forms induced by right multiplication of the group $SO(6)$ on it self).

To obtain the adjoint representation $M_b{}^a$ in terms of the fundamental representation $M^n{}_m$ we define a right coinvariant Maurer-Cartan 1-form ω on $SO_q(6)$ as

$$\omega^n{}_k = dM^n{}_m S\left(M^m{}_k\right) \tag{89}$$

and

$$\tilde{\theta}^n{}_k = S\left(M^n{}_m\right)dM^m{}_k. \tag{90}$$

The left coinvariant Maurer-Cartan 1-forms $\tilde{\theta}$ and the right coinvariant Maurer-Cartan 1-form ω are related by

$$\begin{aligned}\tilde{\theta}^i{}_j &= S\left(M^i{}_m\right)\omega^m{}_k M^k{}_j \\ \omega^i{}_j &= M^i{}_m\theta^m{}_k S\left(M^k{}_j\right).\end{aligned} \tag{91}$$

Now we calculate the right coaction on $\tilde{\theta}^i{}_j$ by expressing them in terms of $\omega^i{}_j$, using the homomorphism property of the right coaction, the right coinvariance of $\omega^i{}_j$ and translating the latter back into the $\tilde{\theta}^i{}_j$. The result is

$$\begin{aligned}\Delta_R\left(\tilde{\theta}^i{}_j\right) &= \tilde{\theta}^l{}_n \otimes S\left(M^i{}_l\right)M^n{}_j \\ &= \tilde{\theta}^l{}_n \otimes M_l{}^{ni}{}_j\end{aligned} \tag{92}$$

[12] We recall that the $q = 1$ adjoint representation is defined as: $x^{-1}T_b x \equiv M_b{}^a(x)\,T_a$ where the infinitesimal operators carry the adjoint representation of the Lie group $SO(6)$.

with
$$M_l{}^{ni}{}_j = S\left(M^i{}_l\right) M^n{}_j. \tag{93}$$
If we replace the index pairs ${}^i{}_j$ with the upper indices (row indices) $a = 1, ..., 36$ and the index pairs ${}_l{}^n$ with lower indices (column indices) b, Equ. (92) gives
$$\Delta_R\left(\tilde{\theta}^a\right) \tilde{\theta}^b \otimes M_b{}^a. \tag{94}$$
The co-structures on $M_b{}^a$ are given by
$$\begin{aligned}
\Delta\left(M_b{}^a\right) &= M_b{}^c \otimes M_c{}^a, \\
\epsilon\left(M_b{}^a\right) &= \delta^a_b, \\
S\left(M_b{}^c\right) M_c{}^a &= M_b{}^c M_c{}^a = \delta^a_b.
\end{aligned} \tag{95}$$

In the quantum case we have $\tilde{\theta}^i{}_j M^n{}_m \neq M^n{}_m \tilde{\theta}^i{}_j$ in general, the bimodule structure of Γ being non-trivial for $q \neq 1$. There exist linear functionals $f^i{}_{jq}{}^p : Fun\left(SO_q\left(6\right)\right) \to \mathbb{C}$ for these left invariant basis such that

$$\begin{aligned}
\tilde{\theta}^i{}_j M^n{}_m &= \left(f^i{}_{jp}{}^q \star M^n_m\right) \tilde{\theta}^p{}_q = \left(id \otimes f^i{}_{jp}{}^q\right) \Delta\left(M^n{}_m\right) \tilde{\theta}^p_q \\
&= M^n{}_l\, f^i{}_{jp}{}^q \left(M^l{}_m\right) \tilde{\theta}^p{}_q,
\end{aligned} \tag{96}$$

$$M^n{}_m \tilde{\theta}^i{}_j = \tilde{\theta}^p{}_q \left[\left(f^i{}_{jp}{}^q \circ S\right) \star M^n{}_m\right]. \tag{97}$$

The functionals $f^i{}_{jp}{}^q$ are uniquely determined by Equ. (96) and satisfy

$$f^i{}_{jp}{}^q \left(M^n{}_m M^l{}_k\right) = f^i{}_{jr}{}^s \left(M^n{}_m\right) f^r{}_{sp}{}^q \left(M^l{}_k\right) \tag{98}$$
$$\left(f^i{}_{jp}{}^q \star M^n{}_m\right) M_r{}^{sp}{}_q = M_q{}^{pi}{}_j \left(M^n{}_m \star f^q{}_{pr}{}^s\right). \tag{99}$$

This result places significant constraints on the possible bicovariant calculi which are consistent with the assumption that the differentials of the generators should generate the bimodules of forms as a left $\mathcal{A}$-module. In fact we are passing directly to a class of quotients of the bimodule Γ which we then constraint by the requirement that the bicovariance is not destroyed.
We can define a new basis

$$\tilde{\theta}^{nm} = \tilde{\theta}^m{}_j C^{jn}, \qquad \tilde{\theta}^m{}_n = \tilde{\theta}^{mj} C_{jn}. \tag{100}$$

The functionals f^{ml}_{ij} corresponding to the basis $\tilde{\theta}^{mn}$ are given by

$$f^{ml}{}_{ij} = C^{pl} f^m{}_{pi}{}^q C_{jq}. \tag{101}$$

The relation between the left and the right multiplication for this basis is

$$\tilde{\theta}^{mn} M^i{}_j = \left(f^{mn}{}_{pq} \star M^i{}_j \right) \tilde{\theta}^{pq}. \tag{102}$$

The exterior derivative d is defined as

$$\begin{aligned} dM^n{}_m &= \frac{1}{\mathcal{N}} \left[\tilde{X}, M^n{}_m \right] = \left(\tilde{\chi}_i{}^j \star M^n{}_m \right) \tilde{\theta}^i{}_j \\ &= \left(id \otimes \tilde{\chi}_i{}^j \right) \Delta \left(M^n{}_m \right) \tilde{\theta}^i{}_j = \tilde{\chi}_i{}^j \left(M^k{}_m \right) M^n{}_k \tilde{\theta}^i{}_j, \end{aligned} \tag{103}$$

where $\tilde{X} = \tilde{\theta}^i{}_i$ is the singlet representation of $\tilde{\theta}^i{}_j$, $\mathcal{N} \in \mathbb{C}$ is the normalization constant and $\tilde{\chi}_i{}^j$ are left-invariant vector fields.
Now, let us find the relation between the left Cartan-Maurer forms $\tilde{\theta}^i{}_j$ and the forms obtained by taking the tensor product $\theta^{ab} = \theta^a \bar{\theta}^b$.
From Equ. (90) and Equ. (66) we get

$$\tilde{\theta}^i{}_j = \chi_{ab} \left(M^i{}_j \right) \theta^{ab}. \tag{104}$$

Equ. (64) gives

$$\chi_{ab} \left(M^i{}_j \right) \theta^{ab} M^n{}_m = f^{ab}_{Ad \quad cd} \left(M^k{}_m \right) M^n{}_k \chi_{ab} \left(M^i{}_j \right) \theta^{cd}. \tag{105}$$

Comparing with Equ. (96) we find

$$f^{cd}_{Ad \quad ab} \left(M^k{}_m \right) \chi_{cd} \left(M^i{}_j \right) = f^i{}_{jp}{}^q \left(M^k{}_m \right) \chi_{ab} \left(M^p{}_q \right). \tag{106}$$

This is the first identity. The second identity gives a relation between left-invariant vector fields. In fact, from Equ. (66) and Equ. (103) we get

$$\begin{aligned} dM^n{}_m &= M^n{}_k \chi_{ab} \left(M^k{}_m \right) \theta^{ab} \\ &= \tilde{\chi}_i{}^j \left(M^k{}_m \right) M^n{}_k \tilde{\theta}^i{}_j = \tilde{\chi}_i{}^j \left(M^k{}_m \right) M^n{}_k \chi_{ab} \left(M^i{}_j \right) \theta^{ab}, \end{aligned} \tag{107}$$

which gives

$$\chi_{ab} \left(M^k{}_m \right) = \tilde{\chi}_i{}^j \left(M^k{}_m \right) \chi_{ab} \left(M^i{}_j \right). \tag{108}$$

Taking $\tilde{\chi}_i{}^j \left(M^k{}_m \right) = \delta^j_m \delta^k_i$ we find an identity.

6 Quantum Gauge Theories

We formulate the q-deformed gauge theories using the concept of the quantum fiber bundles [15, 17]. We consider a quantum vector $E(X_B, V, \mathcal{A})$ with a noncommutative algebra base X_B (the quantum spacetime), a comodule algebra V considered as a fiber of E and a structure quantum group $\mathcal{A}$ playing the role of the quantum symmetry group. The matter fields (sections) ψ are maps: $V \to X_B$. The quantum gauge transformations T are defined in terms of the right coaction and act on the sections as

$$\begin{aligned}
\psi'^i &= (\psi \star T)\left(\theta^i\right) = (\psi \otimes T)\,\Delta_R\left(\theta^i\right) \\
&= \psi\left(\theta^j\right) \otimes T\left(S\left(M^i{}_j\right)\right) = \psi\left(\theta^j\right) \otimes T^{-1}\left(M^i{}_j\right), \qquad (109)
\end{aligned}$$

where $\theta^i \in V$ and $T : \mathcal{A} \to X_B$ is a convolution invertible map such that $T(1_{\mathcal{A}}) = 1_{X_B}$.

For any two quantum gauge transformations T and T', the convolution product is defined as

$$(T \star T')\left(M^i{}_j\right) = (T \otimes T')\,\Delta\left(M^i{}_j\right). \qquad (110)$$

The quantum inverse gauge transformation is defined as

$$T^{-1i}{}_j = T\left(S\left(M^i{}_j\right)\right). \qquad (111)$$

Using the right covariance property of the quantum $\mathcal{A}$-bimodule V, we get

$$\begin{aligned}
\psi''^i &= (\psi' \star T)\left(\theta^i\right) = ((\psi \star T)\star)\left(\theta^i\right) \\
&= (\psi \otimes T \otimes T)\,(id \otimes \Delta)\,\Delta_R\left(\theta^i\right) \\
&= (\psi \star (T \star T))\left(\theta^i\right) \qquad (112)
\end{aligned}$$

which simply reflect the closure of the finite quantum gauge transformations. Let us now define a covariant exterior derivative as a linear map on the set of sections $\Gamma(E)$, $\nabla : \Gamma(E) \to \Gamma^1(E)$, where $\Gamma^1(E)$ is the set of one-form sections. We require that these sections transform with the same rule as the corresponding matter fields, i.e. :

$$\begin{aligned}
\nabla'^n{}_m &= (\nabla \otimes T)\,Ad_R\left(M^n{}_m\right) \\
&= \nabla^k{}_l \otimes T^{-1n}{}_k T^l{}_m, \qquad (113)
\end{aligned}$$

which gives

$$\begin{aligned}
\nabla'^n{}_m \psi'^m &= \left(\nabla^k{}_l \otimes T^{-1n}_{\;\;k} T^l{}_m\right)\left(\psi^j \otimes T^{-1m}_{\;\;j}\right) \\
&= \nabla^k{}_j \psi^j \otimes T^{-1n}{}_k. \qquad (114)
\end{aligned}$$

The quantum exterior derivative d on the base space X_B is defined in terms of the covariant derivative

$$\nabla^n{}_m = d\delta^n_m + A^n{}_m \tag{115}$$

where $A^n{}_m$ are the quantum Lie-algebra valued matrices of one forms on X_B, i.e. $A^n{}_m = A^{ab}\chi_{ab}\left(M^n{}_m\right)$.

The consistency of Equ. (113) requires that $A^n{}_m$ transforms as

$$A'^n{}_m = A^k{}_l \otimes T^{-1n}{}_k T^l{}_m + \delta^k_l \otimes T^{-1n}{}_k dT^l{}_m, \tag{116}$$

and two successive transformations act as

$$\begin{aligned} A''^n{}_m &= A'^k{}_l \otimes T^{-1n}{}_k T^l{}_m + \delta^k_l \otimes T^{-1n}{}_k dT^l{}_m \\ &= A^p{}_q \otimes T^{-1k}{}_p T^q{}_l \otimes T^{-1n}{}_k T^l{}_m \\ &\quad +1 \otimes T^{-1k}{}_p dT^p_l \otimes T^{-1n}{}_k T^l{}_m + 1 \otimes T^{-1n}{}_p dT^p_m \\ &= A^p{}_q \otimes \left(T^n_k \otimes T^k_p\right)^{-1}\left(T^q{}_l \otimes T^l{}_m\right) \\ &\quad +1 \otimes \left(T^{-1k}{}_p \otimes T^{-1n}{}_k\right)\left(dT^p_l \otimes T^l{}_m\right) + 1 \otimes T^{-1n}{}_p dT^p{}_m \\ &= A^p{}_q \otimes \left(T^n_k \otimes T^k_p\right)^{-1}\left(T^q{}_l \otimes T^l{}_m\right) \\ &\quad +1 \otimes \left(T^n_k \otimes T^k_p\right)^{-1} d\left(T^p{}_l \otimes T^l{}_m\right). \end{aligned} \tag{117}$$

This equation shows that if T^n_m is a gauge transformation on the connection $A^p{}_q$, then $T'^n_m = (T \star T)\left(M^n{}_m\right) = (T \otimes T)\,\Delta\left(M^n{}_m\right)$.

As in the undeformed case, the quantum two-form curvature associated to the connection is given by

$$F^n{}_m = \nabla^n{}_k \wedge \nabla^k{}_m = dA^n{}_m + A^n{}_k \wedge A^k{}_m, \tag{118}$$

and transforms as

$$\begin{aligned} F'^n_m &= \nabla'^n{}_i \wedge \nabla'^i{}_m = \left(\nabla^j{}_l \otimes T^{-1n}{}_j T^l{}_i\right)\left(\nabla^k{}_p \otimes T^{-1i}{}_k T^p{}_m\right) \\ &= F^j{}_k \otimes T^{-1n}{}_j T^k_m. \end{aligned} \tag{119}$$

We can also express this equation in terms of the right adjoint coaction as

$$F'^n{}_m = F'\left(M^n{}_m\right) Ad_R\left(M^n{}_m\right) = (F \wedge T)\left(M^n{}_m\right). \tag{120}$$

The closure of the gauge transformations for the curvature can be obtained as

$$\begin{aligned} F''\left(M^n{}_m\right) &= \left(F' \wedge T\right)\left(M^n{}_m\right) = \left((F \wedge T) \wedge T\right)\left(M^n{}_m\right) \\ &= F^i{}_j \otimes \left(T^n_k \otimes T^k_i\right)^{-1}\left(T^j{}_l \otimes T^l{}_m\right). \end{aligned} \tag{121}$$

We can decompose the curvature in terms of left-invariant vector basis as

$$\begin{aligned} F^n{}_m &= F^{ab}\chi_{ab}\left(M^n{}_m\right) \\ &= dA^{ab}\chi_{ab}\left(M^n{}_m\right) + A^{ab}\wedge A^{cd}\chi_{ab}\left(M^n{}_p\right)\chi_{cd}\left(M^p{}_m\right). \end{aligned} \tag{122}$$

We define the infinitesimal variations around unity of the gauge transformations as

$$\begin{aligned} \delta_\alpha T^n{}_m &= (T\star\alpha)\left(M^n{}_m\right) = \left(T\star\alpha^{ab}\chi_{ab}\right)\Delta\left(M^n{}_m\right) \\ &= T^n_k\otimes\alpha^{ab}\chi_{ab}\left(M^k{}_m\right), \end{aligned} \tag{123}$$

where α are infinitesimal quantum gauge parameters of the transformation T. The infinitesimal variation corresponding to α is given by

$$\alpha'\left(M^n{}_m\right) = (\alpha\otimes T)\,Ad_R\left(M^n{}_m\right). \tag{124}$$

The infinitesimal variation of the connection is

$$\delta_\alpha A^{ab} = -1\otimes d\alpha^{ab} + A^{cd}\otimes\alpha^{ef}C_{cdef}{}^{ab}, \tag{125}$$

where $C_{cdef}{}^{ab}$ are the quantum structure constants defined in Equ. (80).

The curvature transforms with the right adjoint coaction. Its infinitesimal gauge transformation reads

$$\delta_\alpha F^n{}_m = (F\otimes\alpha)\,Ad_R\left(M^n{}_m\right). \tag{126}$$

In terms of components, we find

$$\delta_\alpha F^{ab} = F^{cd}\otimes\alpha^{ef}C_{cdef}{}^{ab}. \tag{127}$$

To illustrate the construction of BRST and anti-BRST transformations, let us consider the simple example of BF-Yang-Mills theories. These theories supply a complete nonperturbative Nicolai map for Yang-Mills theory on any Riemann surface which reduces the partition function to an integral over the moduli space of flat connections, with measure given by the Ray-Singer torsion.

The quantum Lagrangian describing these models (hereafter, the space-time indices are omitted for simplicity) can be written as

$$L_{BFYM} = < iB^{ab}F^{cd} + g^2 B^{ab}B^{cd} > g_{abcd} \tag{128}$$

where B is a quantum Lie-algebra valued 2-form, g^2 is the coupling constant and g_{abcd} is the quantum Killing metric defined in Equ. (83). The quantum

Lie-algebra valued curvature $F : \mathcal{A} \to \Gamma^2(X_B)$ is given by Equ. (118).
The quantum BRST transformations for these models are obtained, as usual, by replacing the quantum infinitesimal parameters by the ghosts:

$$\begin{aligned}
\delta A &= -dc^{ab}\chi_{ab} - A^{ab} \cdot c^{cd} (\chi_{ab} \otimes \chi_{cd}) Ad_R, \\
\delta F &= -F^{ab} \cdot c^{cd} (\chi_{ab} \otimes \chi_{cd}) Ad_R, \\
\delta B &= -B^{ab} \cdot c^{cd} (\chi_{ab} \otimes \chi_{cd}) Ad_R, \\
\delta c &= -\frac{1}{2} c^{ab} \cdot c^{cd} (\chi_{ab} \otimes \chi_{cd}) Ad_R, \\
\delta \overline{c} &= b, \quad \delta b = 0
\end{aligned} \tag{129}$$

and the corresponding quantum anti-BRST transformations as

$$\begin{aligned}
\overline{\delta} A &= -d\overline{c}^{ab}\chi_{ab} - A^{ab} \cdot c^{cd} (\chi_{ab} \otimes \chi_{cd}) Ad_R, \\
\overline{\delta} F &= -F^{ab} \cdot \overline{c}^{cd} (\chi_{ab} \otimes \chi_{cd}) Ad_R, \\
\overline{\delta} B &= -B^{ab} \cdot \overline{c}^{cd} (\chi_{ab} \otimes \chi_{cd}) Ad_R, \\
\overline{\delta} c &= -b - \frac{1}{2} c^{ab} \cdot c^{cd} (\chi_{ab} \otimes \chi_{cd}) Ad_R, \\
\overline{\delta} \overline{c} &= -\frac{1}{2} \overline{c}^{ab} \cdot \overline{c}^{cd} (\chi_{ab} \otimes \chi_{cd}) Ad_R, \\
\overline{\delta} b &= -b^{ab} \cdot \overline{c}^{cd} (\chi_{ab} \otimes \chi_{cd}) Ad_R.
\end{aligned} \tag{130}$$

A straightforward but tedious calculation using essentially the quantum Jacobi identity shows that the quantum BFYM action is separately invariant under these quantum BRST and anti-BRST transformations. Using the quantum BRST transformations we also find the quantum analogue of the Taylor-Slavnov identity.

To define the quantum analogue of the Batalin-Vilkovisky operator [50] we have to introduce the quantum left and right functional partial derivatives. Given a quantum field $\varphi = \varphi^{ab}\chi_{ab}$, $\varphi^* = \varphi^{*ab}\chi_{ab}$ and a quantum functional F we define

$$\frac{d}{dt}|_{t=0}\ F\left(\varphi^{ab} + t\varrho^{ab}\right) = \int_{X_B} \left\langle \varrho^{ab}, \frac{\overrightarrow{\partial} F}{\partial \varphi_{cd}} \right\rangle g_{abcd} = \int_{X_B} \left\langle \frac{F\overleftarrow{\partial}}{\partial \varphi_{ab}}, \varrho^{cd} \right\rangle g_{abcd} \tag{131}$$

and the quantum Batalin Vilkovisky antibracket of two functionals F, G:

$$(F,G)=\int_{X_B}\left\langle\frac{F\overleftarrow{\partial}}{\partial\varphi^i_{ab}},\frac{\overrightarrow{\partial}G}{\partial\varphi^+_{i\,cd}}\right\rangle g_{abcd}-(-1)^{\deg\varphi^{i\,ab}}\left\langle\frac{F\overleftarrow{\partial}}{\partial\varphi^{+\,i}_{ab}},\frac{\overrightarrow{\partial}G}{\partial\varphi_{i\,cd}}\right\rangle g_{abcd}. \tag{132}$$

We can construct a new action called the quantum BRST action defined by

$$S=\int_{X_B}d^4_qx\left(L_{BFYM}+(-)^{\epsilon_i}\,\varphi^{*\,ab}_i\,\delta\,\varphi^{i\,cd}g_{abcd}\right), \tag{133}$$

where we have used the quantum BRST transformations δ defined in Equ. (129). ϵ_i is the Grassmann parity of φ^i and g_{abcd} is the quantum Killing metric. We have denoted all the antifields by $\varphi^{*\,i\,ab}$ and the fields and the ghosts by $\varphi^{i\,ab}$.
By construction the quantum BRST action is such:

$$\begin{aligned}\frac{\overrightarrow{\partial}S}{\delta\varphi^{*\,i\,ab}}&=-\delta\varphi^{i\,cd}\,g_{abcd}=-\left(S,\varphi^{i\,cd}\right)g_{abcd}\\ \frac{\overrightarrow{\partial}S}{\delta\varphi^{i\,ab}}&=-\delta\varphi^{*\,i\,cd}\,g_{cdab}=-\left(S,\varphi^{*\,i\,cd}\right)g_{cdab}.\end{aligned} \tag{134}$$

These quantum functionals derivatives satisfy the quantum Jacobi identities. Furthermore, the quantum action S satisfies the quantum master equation

$$(S,S)=0, \tag{135}$$

which is a direct consequence of $\delta^2=0$.
Now, we define the quantum Batalin Vilkovisky operator as

$$\Delta=(-)^{\epsilon_i}\frac{\overrightarrow{\partial}}{\partial\varphi^i_{ab}}\frac{\overrightarrow{\partial}}{\partial\varphi^*_{i\,cd}}g_{abcd}. \tag{136}$$

The quantum Batalin Vilkovisky operator coincides with the usual one in the limit $q=1$.

7 A Map between q-Gauge Fields and Ordinary Gauge Fields

The starting point for this investigation is the wish to define the analogue, in the quantum group picture [24], of the Seiberg Witten map [32] argued using the ideas of noncommutative geometry [51]. In the framework of string theory Seiberg and Witten have noticed that the noncommutativity depends on the choice of the regularization procedure: it appears in point-spliting regularization whereas it is not present in the Pauli Villars regularization. This observation led them to argue that there exists a map connecting the noncommutative gauge fields and gauge transformation parameter to the ordinary gauge fields and gauge parameter. This map can be interpreted as an expansion of the noncommutative gauge field in θ. Along similar lines, we have introduced in Refs [28, 29, 30] a new map between the q-deformed and undeformed gauge theories. This map can be seen as an infinitesimal shift in the parameter q, and thus as an expansion of the deformed gauge fields in q.

To begin we consider the undeformed action

$$S = -\frac{1}{4}\int d^4x\ F_{\mu\nu}F^{\mu\nu}, \tag{137}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{138}$$

S is invariant with respect to infinitesimal gauge transformation:

$$\delta_\lambda A_\mu = \partial_\mu \lambda. \tag{139}$$

Now let us study the quantum gauge theory on the quantum plane $\hat{x}\hat{y} = q\hat{y}\hat{x}$. In general, the product of functions on a deformed space is defined via the Gerstenhaber star product [33]: Let $\mathcal{A}$ be an associative algebra and let $D_i, E^i : \mathcal{A} \to \mathcal{A}$ be a pairwise derivations.
Then the star product of a and b is given by

$$a \star b = \mu \circ e^{\zeta \sum_i D_i \otimes E^i} a \otimes b, \tag{140}$$

where ζ is a parameter and μ the undeformed product given by

$$\mu\,(f \otimes g) = fg. \tag{141}$$

On the Manin plane $\hat{x}\hat{y} = q\hat{y}\hat{x}$, we can write this star product as:

$$f \star g = \mu \circ e^{\frac{i\eta}{2}\left(x\frac{\partial}{\partial x}\otimes y\frac{\partial}{\partial y} - y\frac{\partial}{\partial y}\otimes x\frac{\partial}{\partial x}\right)} f \otimes g \tag{142}$$

A straightforward computation gives then the following commutation relations

$$x \star y = e^{\frac{i\eta}{2}} xy, \qquad y \star x = e^{\frac{-i\eta}{2}} yx. \tag{143}$$

Whence

$$x \star y = e^{i\eta} y \star x, \qquad q = e^{i\eta}. \tag{144}$$

Thus we recover the commutation relations for the Manin plane:
$\hat{x}\hat{y} = q\hat{y}\hat{x}$.
We can also write the product of functions as

$$f \star g = f e^{\frac{i}{2}\overleftarrow{\partial}_k \theta^{kl}(x,y)\overrightarrow{\partial}_l} g \tag{145}$$

where the antisymmetric matrix $\theta^{kl}(x,y) = \eta xy\epsilon^{kl}$
with $\epsilon^{12} = -\epsilon^{21} = 1$.

Expanding to first nontrivial order in η, we find

$$f \star g = fg + \frac{i}{2}\eta xy\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right). \tag{146}$$

The q-deformed infinitesimal gauge transformations are defined by

$$\begin{aligned} \widehat{\delta}_{\widehat{\lambda}}\widehat{A}_\mu &= \partial_\mu\widehat{\lambda} + i\left[\widehat{\alpha}, \widehat{A}_\mu\right]_\star = \partial_\mu\widehat{\lambda} + i\widehat{\lambda}\star\widehat{A}_\mu - i\widehat{A}_\mu\star\widehat{\lambda}, \\ \widehat{\delta}_{\widehat{\lambda}}\widehat{F}_{\mu\nu} &= i\widehat{\lambda}\star\widehat{F}_{\mu\nu} - i\widehat{F}_{\mu\nu}\star\widehat{\lambda}. \end{aligned} \tag{147}$$

To first order in $\theta(x,y)$, the above formulas for the gauge transformations read

$$\begin{aligned} \widehat{\delta}_{\widehat{\lambda}}\widehat{A}_\mu &= \partial_\mu\widehat{\lambda} - \frac{1}{2}\theta^{\rho\sigma}(x,y)\left(\partial_\rho\lambda\partial_\sigma A_\mu - \partial_\rho A_\mu\partial_\sigma\lambda\right) \\ \widehat{\delta}_{\widehat{\lambda}}\widehat{F}_{\mu\nu} &= -\frac{1}{2}\theta^{\rho\sigma}(x,y)\left(\partial_\rho\lambda\partial_\sigma F_{\mu\nu} - \partial_\rho F_{\mu\nu}\partial_\sigma\lambda\right). \end{aligned} \tag{148}$$

To ensure that an ordinary gauge transformation of A by λ is equivalent to q-deformed gauge transformation of $\widehat{A}$ by $\widehat{\lambda}$ we consider the following relation [32]

$$\widehat{A}(A) + \widehat{\delta}_{\widehat{\lambda}}\widehat{A}(A) = \widehat{A}(A + \delta_\lambda A) \tag{149}$$

We first work the first order in θ

$$\begin{aligned} \widehat{A} &= A + A'(A) \\ \widehat{\lambda}(\lambda, A) &= \lambda + \lambda'(\lambda, A). \end{aligned} \tag{150}$$

Expanding in powers of θ we find

$$A'_\mu (A + \delta_\lambda A) - A'_\mu (A) - \partial_\mu \lambda' = \theta^{kl}(x, y)\, \partial_k A_\mu \partial_l \lambda \tag{151}$$

The solutions are given by

$$\widehat{A}_\mu = A_\mu - \frac{1}{2}\theta^{\rho\sigma}(x, y)\left(A_\rho F_{\sigma\mu} + A_\rho \partial_\sigma A_\mu\right), \tag{152}$$

$$\widehat{\lambda} = \lambda + \frac{1}{2}\theta^{\rho\sigma}(x, y)\, A_\sigma \partial_\rho \lambda. \tag{153}$$

The q-deformed curvature $\widehat{F}_{\mu\nu}$ is given by

$$\begin{aligned} \widehat{F}_{\mu\nu} &= \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - i\left[\widehat{A}_\mu, \widehat{A}_\nu\right]_\star \\ &= \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - i\widehat{A}_\mu \star \widehat{A}_\nu + i\widehat{A}_\nu \star \widehat{A}_\mu. \end{aligned} \tag{154}$$

Finally, we find

$$\begin{aligned} \widehat{F}_{\mu\nu} = {} & F_{\mu\nu} + \theta^{\rho\sigma}(x)\left(F_{\mu\rho}F_{\nu\sigma} - A_\rho \partial_\sigma F_{\mu\nu}\right) \\ & -\frac{1}{2}\partial_\mu \theta^{\rho\sigma}(x)\left(A_\rho F_{\sigma\nu} + A_\rho \partial_\sigma A_\nu\right) && (155) \\ & +\frac{1}{2}\partial_\nu \theta^{\rho\sigma}(x)\left(A_\rho F_{\sigma\mu} + A_\rho \partial_\sigma A_\mu\right), && (156) \end{aligned}$$

which we can write as

$$\widehat{F}_{\mu\nu} = F_{\mu\nu} + f_{\mu\nu} + o\left(\eta^2\right), \tag{157}$$

where $f_{\mu\nu}$ is the quantum correction linear in η. The quantum analogue of Equ. (137) is given by

$$\widehat{S} = -\frac{1}{4}\int d^4x\ \widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu}. \tag{158}$$

We can easily see from this equation that the q-deformed action contains non-renormalizable vertices of dimension six. Other term which are proportional to $\partial_\mu \theta^{\rho\sigma}(x)$ appear.

Finally, let us emphasize once more that we can also consider a quantum gauge theory with a quantum gauge group as a symmetry group defined on a quantum space. This gives a general map between deformed and ordinary gauge fields [28].

8 The Quantum Anti-de Sitter Space

In this section, we propose to construct the metrics of the quantum analogue of the classical anti-de Sitter space AdS_5.
The AdS_5 space is a 5-dimensional manifold with constant curvature and signature (+,-,-,-,-). It can be embedded as an hyperboloid into a 6-dimensional flat space with signature (+,+,-,-,-,-), by

$$z_0^2 + z_5^2 - z_1^2 - z_2^2 - z_3^2 - z_4^2 = R^2, \tag{159}$$

where R will be called the "radius" of the AdS_5 space.

To define the quantum anti-de Sitter space we follow the method of Ref. [52] used for AdS_4^q. The quantum anti-de Sitter space AdS_5^q is nothing but the quantum sphere S_5^q with a suitable reality structure. For $|q| = 1$ we consider the conjugation[13][19] defined as $M^\times = M$. The unique associated quantum space conjugation is $(x^a)^\times = x^a$. By this conjugation on the quantum orthogonal we cannot get the desired quantum AdS space. We introduce another operation on the quantum orthogonal group as

$$M^\dagger = DMD^{-1} \tag{160}$$

where the matrix D is given by

$$D = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}. \tag{161}$$

We can easily prove that the D matrix is a special element of the quantum orthogonal group [53]. The quantum AdS group is obtained by the combined

[13] Let us recall that a $*$-structure or $*$-conjugation on a Hopf algebra $\mathcal{A}$ is an anti-automorphism $(\eta ab)^* = \bar{\eta} b^* a^* \quad \forall a, b \in \mathcal{A}, \quad \forall \eta \in \mathbb{C}$; coalgebra automorphism $\Delta \circ * = (* \otimes *) \circ \Delta, \quad \epsilon \circ * = \epsilon$ and involution $*^2 = id$. It follows that $* \circ S^{-1} = S \circ *$ i.e. $\left[S^{-1}\left(M^n{}_m\right)\right]^* = S\left(M^{*n}{}_m\right)$.

operation $M^* \equiv M^{\times\dagger} = DMD^{-1}$. The induced conjugation on the quantum space is $x^* \equiv x^{\times\dagger} = Dx$. We can check that the conjugation really gives the quantum AdS group and quantum AdS space. We should find a linear transformation $x \to x' = Ux, \quad M \to M' = UMU^{-1}$ such that the new coordinates x' and M' are real and the new metric $C' = (U^{-1})^t CU^{-1}$ diagonal in the $q \to 1$ limit, $C'|_{q=1} = diag\,(1,-1,-1,-1,-1,1)$. The metric C is defined in Equ. (45). We find the following U matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & i & i & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{162}$$

and the corresponding quantum metric is

$$C' = \begin{pmatrix} \frac{1}{2}q^2 + \frac{1}{2q^2} & -\frac{1}{2}q^2 + \frac{1}{2q^2} & 0 & 0 & 0 & 0 \\ \frac{1}{2}q^2 - \frac{1}{2q^2} & -\frac{1}{2}q^2 - \frac{1}{2q^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}q - \frac{1}{2q} & \frac{1}{2}q - \frac{1}{2q} \\ 0 & 0 & 0 & 0 & -\frac{1}{2}q + \frac{1}{2q} & \frac{1}{2}q + \frac{1}{2q} \end{pmatrix}. \tag{163}$$

For q real we consider the second conjugation given in [19] and realized via the metric, i.e. $M^* = C^t M C^t$. The metric C is defined in Equ. (45). The condition on the braiding R matrix is: $\overline{R} = R$. To get the quantum AdS group and the AdS space we have to consider another operation on the quantum orthogonal space as:

$$M^{\ddagger} = AMA^{-1} \tag{164}$$

where the matrix A is given by

$$A = \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ & & & & & 1 \end{pmatrix}. \tag{165}$$

We obtain the AdS quantum group by using the conjugation $M^{*\ddagger} = AC^t M C^t A^{-1}$. The induced conjugation on the quantum space is $x^{*\ddagger} =$

$C^t Ax$. To prove that this combination really gives the quantum AdS group and quantum AdS space we should find a linear transformation $x \to x' = Vx, \quad M \to M' = VMV^{-1}$. Such that the new coordinates x' and M' are real and the new metric $C'|_{q=1} = diag\,(1,-1,-1,-1,-1,1)$. We find the following V matrix

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & q^2 \\ 0 & -i & 0 & 0 & -iq & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & i & i & 0 & 0 \\ 0 & q^{-1} & 0 & 0 & -1 & 0 \\ iq^{-2} & 0 & 0 & 0 & 0 & -i \end{pmatrix} \tag{166}$$

and the quantum metric C' is given by

$$C' = \begin{pmatrix} \frac{1}{2}+\frac{1}{2q^4} & 0 & 0 & 0 & 0 & -\frac{1}{2}iq^2+\frac{1}{2}\frac{i}{q^2} \\ 0 & -\frac{1}{2}-\frac{1}{2q^2} & 0 & 0 & \frac{1}{2}iq-\frac{1}{2}\frac{i}{q} & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -\frac{1}{2}iq+\frac{1}{2}\frac{i}{q} & 0 & 0 & -\frac{1}{2}-\frac{1}{2}q^2 & 0 \\ \frac{1}{2}iq^2-\frac{1}{2}\frac{i}{q^2} & 0 & 0 & 0 & 0 & \frac{1}{2}q^4+\frac{1}{2} \end{pmatrix}. \tag{167}$$

These metrics can be used in the definition of Lagrangians defined on the quantum Anti-de Sitter space AdS_5^q. As an example of such a theory is the quantum Chern-Simons term which is present in the low energy effective action of type IIB superstring theory compactified on the quantum anti-de Sitter space.

9 q-deformed conformal correlation functions

In this section, we construct the q-deformed two- and three- point conformal correlation functions in field theories that are assumed to possess an invariance under a quantum deformation of $SO\,(4,2)$. In the course of these investigations we rely on the general formalism developed by Dobrev [54] who first introduced the q-deformation of $D = 4$ conformal algebra [14] and constructed its q-difference realizations. Let us recall that the positive energy irreducible representations of $so\,(4,2)$ are labelled by the lowest value of the energy E_0, the spin $s_0 = j_1 + j_2$ and by the helicity $h_0 = j_1 - j_2$, and

[14] This quantum algebra was also studied, and in addition its contraction to deformed Poincaré algebra given in Ref. [55].

these are eigenvalues of a Cartan subalgebra $\mathcal{H}$ of $so\,(4,2)$. We shall label the representations of $U_q\,(so\,(4,2))$ in the same way and thus we shall take for $U_q\,(so\,(4,2))$ and its complexification $U_q\,(so\,(6,\mathbb{C}))$ the same Cartan subalgebra. We recall that the q-deformation $U_q\,(so\,(6,\mathbb{C}))$ is defined [24, 56] as the associative algebra over $\mathbb{C}$ with Chevalley generators $X_j^{\pm}$, H_j, $j=1,2,3$.

The Cartan-Chevalley basis of $U_q\,(sl\,(4,\mathbb{C}))$ is given by the formulae:

$$\begin{aligned} [H_j, H_k] &= 0 \qquad \left[H_j, X_k^{\pm}\right] = \pm\, a_{jk}\, X_k^{\pm} \\ \left[X_j^{+}, X_k^{-}\right] &= \delta_{jk}\frac{q^{H_j}-q^{-H_j}}{q-q^{-1}} = \delta_{jk}\,[H_j]_q\,. \end{aligned} \tag{168}$$

and the q-analogue of the Serre relations

$$\left(X_j^{\pm}\right)^2 X_k^{\pm} - [2]_q\, X_j^{\pm} X_k^{\pm} X_j^{\pm} + X_k^{\pm}\left(X_j^{\pm}\right)^2 = 0, \tag{169}$$

where $(jk) = (12)\,,(21)\,,(23)\,,(32)$ and (a_{jk}) is the Cartan matrix of $so\,(6,\mathbb{C})$ given by $(a_{jk}) = 2\,(\alpha_j,\alpha_k)\,/\,(\alpha_j,\alpha_j)$; $\alpha_1,\alpha_2,\alpha_3$ are the simple roots of length 2 and the non-zero product between the simple roots are: $(\alpha_1,\alpha_2) = (\alpha_2,\alpha_3) = -1$. The quantum number is defined as $[m]_q = \frac{q^m-q^{-m}}{q-q^{-1}}$.

Explicitly the Cartan matrix is given by:

$$(a_{jk}) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{170}$$

The elements H_j span the Cartan subalgebra $\mathcal{H}$ while the elements $X_j^{\pm}$ generate the subalgebra $\mathcal{G}^{\pm}$ in the standard decomposition $\mathcal{G} \equiv so\,(6,\mathbb{C}) = \mathcal{G}^{+} \oplus \mathcal{H} \oplus \mathcal{G}^{-}$. In particular, the Cartan-Weyl generators for the non-simple roots are given by [57]:

$$\begin{aligned} X_{jk}^{\pm} &= \pm q^{\mp 1/2}\left(q^{1/2} X_j^{\pm} X_k^{\pm} - q^{-1/2} X_k^{\pm} X_j^{\pm}\right) \qquad (jk) = (12)\,,(23) \\ X_{13}^{\pm} &= \pm q^{\mp 1/2}\left(q^{1/2} X_1^{\pm} X_{23}^{\pm} - q^{-1/2} X_{23}^{\pm} X_1^{\pm}\right) \\ &= \pm q^{\mp 1/2}\left(q^{1/2} X_{12}^{\pm} X_3^{\pm} - q^{-1/2} X_3^{\pm} X_{12}^{\pm}\right). \end{aligned} \tag{171}$$

All other commutation relations follow from these definitions:

$$\begin{array}{rcll}
\left[X_a^+, X_{ab}^-\right] & = & -q^{H_a}\, X_{a+1b}^- & \left[X_b^+, X_{ab}^-\right] = X_{ab-1}^-\, q^{-H_b} \quad 1 \le a < b \le 3 \\
\left[X_a^-, X_{ab}^+\right] & = & X_{a+1b}^+\, q^{-H_a} & \left[X_b^-, X_{ab}^+\right] = -q^{H_b}\, X_{ab-1}^+ \quad 1 \le a < b \le 3 \\
X_a^\pm X_{ab}^\pm & = & q X_{ab}^\pm X_a^\pm & X_b^\pm X_{ab}^\pm = q^{-1} X_{ab}^\pm X_a^\pm \quad 1 \le a < b \le 3 \\
X_{12}^\pm X_{13}^\pm & = & q X_{13}^\pm X_{12}^\pm & X_{23}^\pm X_{13}^\pm = q^{-1} X_{13}^\pm X_{23}^\pm \\
\left[X_2^\pm, X_{13}^\pm\right] & = & 0 & \left[X_2^\pm, X_{13}^\mp\right] = 0 \\
\left[X_{12}^+, X_{13}^-\right] & = & -q^{2(H_1+H_2)} X_3^- & \left[X_{12}^-, X_{13}^+\right] = X_3^+ q^{-2(H_1+H_2)} \\
\left[X_{23}^+, X_{13}^-\right] & = & X_1^-\, q^{-2(H_2+H_3)} & \left[X_{23}^-, X_{13}^+\right] = -q^{2(H_2+H_3)} X_1^+ \\
\left[X_{12}^\pm, X_{23}^\pm\right] & = & \lambda X_2^\pm X_{13}^\pm & \left[X_{12}^\pm, X_{23}^\mp\right] = -\lambda q^{\pm H_2} X_1^\pm X_3^\mp
\end{array} \tag{172}$$

where $\lambda = q - q^{-1}$.

The dilatation generator is given by

$$D = \frac{1}{2}\left(H_1 + H_3\right) + H_2. \tag{173}$$

The quantum universal algebra $U_q\left(su\left(2,2\right)\right)$ is a Hopf algebra with coproduct defined by:

$$\begin{array}{rcl}
\Delta\left(H_i\right) & = & H_i \otimes 1 + 1 \otimes H_i \\
\Delta\left(X_{\pm i}\right) & = & X_{\pm i} \otimes q^{H_i/2} + q^{-H_i/2} \otimes X_{\pm i}.
\end{array} \tag{174}$$

and antipode and counit defined as

$$\begin{array}{rcl}
S\left(H_i\right) & = & -H_i, \\
S\left(X_{+i}\right) & = & -q X_{+i}, \qquad S\left(X_{-i}\right) = -q^{-1} X_{-i}, \\
\epsilon\left(H_i\right) & = & \epsilon\left(X_{+i}\right) = 0.
\end{array} \tag{175}$$

Now, let us compute the q-deformed 2-point conformal correlation function of scalar quasiprimary (qp) fields, with canonical dimension d_1 and d_2, defined on the q-deformed Minkowski spacetime [15][58].

$$\begin{array}{rcll}
x_\pm v & = & q^{\pm 1} v x_\pm, & x_\pm \overline{v} = q^{\pm 1} \overline{v} x_\pm, \\
\lambda v \overline{v} & = & x_+ x_- - x_- x_+, & \overline{v} v = v \overline{v}, \\
x_\pm & \equiv & x^0 \pm x^3 \qquad v \equiv x^1 - i x^2 & \overline{v} \equiv x^1 + i x^2.
\end{array} \tag{176}$$

[15] Up to Equ. (184) this section follows the paper [58].

The q-Minkowski length is

$$\mathcal{L}_q = x_- x_+ - q^{-1} v\overline{v}. \tag{177}$$

These qp-fields are reduced functions and can be written as formal power series in the q-Minkowski coordinates:

$$\begin{aligned}
\phi &= \phi(Y) = \phi(v, x_-, x_+, \overline{v}) \\
&= \sum_{j,n,l,m \in Z_+} \mu_{jnlm}\, \phi_{j\ n\ l\ m}, \\
\phi_{jnlm} &= v^j x_-^n x_+^l \overline{v}^m.
\end{aligned} \tag{178}$$

Next we introduce the following operators acting on the reduced functions as

$$\begin{aligned}
\widehat{M}_\kappa \phi(Y) &= \sum_{j,n,l,m \in Z_+} \mu_{jnlm}\, \widehat{M}_\kappa\, \phi_{j\ n\ l\ m} \\
T_\kappa \phi(Y) &= \sum_{j,n,l,m \in Z_+} \mu_{jnlm} T_\kappa \phi_{j\ n\ l\ m}
\end{aligned} \tag{179}$$

where $\kappa = \pm, v, \overline{v}$ and the explicit action on $\phi_{j\ n\ l\ m}$ is defined by

$$\begin{aligned}
\widehat{M}_v \phi_{j\ n\ l\ m} &= \phi_{j+1\ n\ l\ m} \\
\widehat{M}_- \phi_{j\ n\ l\ m} &= \phi_{j\ n+1\ l\ m} \\
\widehat{M}_+ \phi_{j\ n\ l\ m} &= \phi_{j\ n\ l+1\ m} \\
\widehat{M}_{\overline{v}} \phi_{j\ n\ l\ m} &= \phi_{j\ n\ l\ m+1} \\
T_v \phi_{j\ n\ l\ m} &= q^j \phi_{j\ n\ l\ m} \\
T_- \phi_{j\ n\ l\ m} &= q^n \phi_{j\ n\ l\ m} \\
T_+ \phi_{j\ n\ l\ m} &= q^l \phi_{j\ n\ l\ m} \\
T_{\overline{v}} \phi_{j\ n\ l\ m} &= q^m \phi_{j\ n\ l\ m}.
\end{aligned} \tag{180}$$

The q-difference operators are defined by

$$\widehat{D}_\kappa \phi = \frac{1}{\lambda} \widehat{M}_\kappa^{-1} \left(T_\kappa - T_\kappa^{-1} \right) \phi. \tag{181}$$

The representation action of $U_q\left(sl\left(4\right)\right)$ on the reduced functions $\phi\left(Y\right)$ of the representation space C^{Λ}, with the signature $\chi = \chi\left(\Lambda\right) = \left(m_1, m_2, m_3\right) = \left(1, 1-d, 1\right)$ and which corresponds to a spinless "scalar" field $\left[d, j_1, j_2\right] = \left[d, 0, 0\right]$ is given by[16] :

$$
\begin{aligned}
\pi\left(k_1\right)\phi_{j\ n\ l\ m} &= q^{(j-n+l-m)/2}\phi_{j\ n\ l\ m},\\
\pi\left(k_2\right)\phi_{j\ n\ l\ m} &= q^{n+(j+m+d)/2}\phi_{j\ n\ l\ m},\\
\pi\left(k_3\right)\phi_{j\ n\ l\ m} &= q^{(-j-n+l+m)/2}\phi_{j\ n\ l\ m},\\
\pi\left(X_{+1}\right)\phi_{j\ n\ l\ m} &= q^{-1+(j-n-l+m)/2}\left[n\right]_q\phi_{j+1\ n-1\ l\ m}\\
&\quad +q^{-1+(j-n+l-m)/2}\left[m\right]_q\phi_{j\ n\ l+1\ m-1},\\
\pi\left(X_{+2}\right)\phi_{j\ n\ l\ m} &= q^{(-j+m)/2}\left[j+n+m+d\right]_q\phi_{j\ n+1\ l\ m}\\
&\quad +q^{d+(j+n+3m)/2}\left[l\right]_q\phi_{j+1\ n\ l-1\ m+1},\\
\pi\left(X_{+3}\right)\phi_{j\ n\ l\ m} &= -q^{-1+(j+n-l-m)/2}\left[j\right]_q\phi_{j-1\ n\ l+1\ m}\\
&\quad -q^{-1+(3j+n-3l-m)/2}\left[n\right]_q\phi_{j\ n-1\ l\ m+1},
\end{aligned}
$$

$$
\begin{aligned}
\pi\left(X_{-1}\right)\phi_{j\ n\ l\ m} &= q^{2+(-j+n-l+m)/2}\left[j\right]_q\phi_{j-1\ n+1\ l\ m}\\
&\quad +q^{2+(j-n-l+m)/2}\left[l\right]_q\phi_{j\ n\ l-1\, m+1},\\
\pi\left(X_{-2}\right)\phi_{j\ n\ l\ m} &= -q^{(j-m)/2}\left[n\right]_q\phi_{j\ n-1\ l\ m},\\
\pi\left(X_{-3}\right)\phi_{j\ n\ l\ m} &= -q^{(-j-3n+l+3m)/2}\left[l\right]_q\phi_{j+1\ n\ l-1\ m}\\
&\quad -q^{(-j-n+l+m)/2}\left[m\right]_q\phi_{j\ n+1\ l\ m-1},
\end{aligned}
\tag{182}
$$

with $k_i = q^{H_i/2}$.

Now let us define $\mathcal{D} = q^D$, where D is the dilatation generator defined in Equ. (173). The representation of this generator on the reduced functions ϕ is given by

$$
\begin{aligned}
\pi\left(\mathcal{D}\right)\phi\left(Y\right) &= \mu_{jnlm}\pi\left(\mathcal{D}\right)\phi_{j\ n\ l\ m}\\
&= q^d\mu_{jnlm}q^{j+n+l+m}\phi_{j\ n\ l\ m} = q^d\phi\left(qY\right).
\end{aligned}
\tag{183}
$$

The coproduct for this operator is given by

$$\Delta\mathcal{D} = \mathcal{D}\otimes\mathcal{D}. \tag{184}$$

[16]The general case is given in Ref. [57]

Now let us calculate two point q-correlation functions by imposing that they are invariant under the action of $U_q\left(sl\left(4,\mathbb{C}\right)\right)$. We denote the q-deformed correlation functions of N quasiprimary fields as

$$\left\langle \phi_1\left(Y_1\right)...\phi_N\left(Y_N\right)\right\rangle_q = {}_q\left\langle 0\left|\phi_{d_1}\left(Y_1\right)...\phi_{d_N}\left(Y_N\right)\right|0\right\rangle_q, \tag{185}$$

where $|0\rangle_q$ is a $U_q\left(sl\left(4,\mathbb{C}\right)\right)$ invariant vacuum such that $\pi\left(\mathcal{D}\right)|0\rangle_q = |0\rangle_q$, $\pi\left(X_{+i}\right)|0\rangle_q = 0$ and also for ${}_q\langle 0|$. The identities for the two-point correlation functions of two quasiprimary fields of the conformal weights d_1, d_2 are

$$\begin{aligned} \Delta\left(\pi\left(\mathcal{D}\right)\right)\left\langle \phi_1\left(Y_1\right)\phi_2\left(Y_2\right)\right\rangle_q &= \left(\pi\left(\mathcal{D}\right)\otimes\pi\left(\mathcal{D}\right)\right)\left\langle \phi_1\left(Y_1\right)\phi_2\left(Y_2\right)\right\rangle \\ &= \left\langle \phi_1\left(Y_1\right)\phi_2\left(Y_2\right)\right\rangle_q \end{aligned} \tag{186}$$

and

$$\begin{aligned} &\Delta\left(\pi\left(X_{\pm i}\right)\right)\left\langle \phi_1\left(Y_1\right)\phi_2\left(Y_2\right)\right\rangle_q = \\ &\left(\pi\left(X_{\pm i}\right)\otimes q^{\pi\left(H_i/2\right)} + q^{-\pi\left(H_i/2\right)}\otimes\pi\left(X_{\pm i}\right)\right). \\ &\left\langle \phi_1\left(Y_1\right)\phi_2\left(Y_2\right)\right\rangle_q = 0. \end{aligned} \tag{187}$$

The q-correlation functions are covariant under dilatation, whereas the remaining identities lead to six q-difference equations.

Let us first note that

$$\begin{aligned} \phi_{j+1\;n-1\;l\;m} &= q^j v\left(x_-\right)^{-1}\;\phi_{j\;n\;l\;m}, \\ \phi_{j\;n\;l+1\;m-1} &= q^m\phi_{j\;n\;l\;m}\;x_+\left(\overline{v}\right)^{-1}, \end{aligned} \tag{188}$$

and so on,

$$\begin{aligned} q^{\pm j/2}\phi\left(v,x_-,x_+,\overline{v}\right) &= \phi\left(q^{\pm 1/2}v,x_-,x_+,\overline{v}\right), \\ q^{\pm n/2}\phi\left(v,x_-,x_+,\overline{v}\right) &= \phi\left(v,q^{\pm 1/2}x_-,x_+,\overline{v}\right),... \end{aligned} \tag{189}$$

and

$$\begin{aligned} \left[n\right]_q\phi &= \lambda^{-1}\left(\phi\left(v,qx_-,x_+,\overline{v}\right)-\phi\left(v,q^{-1}x_-,x_+,\overline{v}\right)\right) \\ &= \widehat{D}_-\,\phi\left(v,x_-,x_+,\overline{v}\right), \\ \left[m\right]_q\phi &= \lambda^{-1}\left(\phi\left(v,x_-,x_+,q\overline{v}\right)-\phi\left(v,x_-,x_+,q^{-1}\overline{v}\right)\right) \\ &= \widehat{D}_{\overline{v}}\,\phi\left(v,x_-,x_+,\overline{v}\right),... \end{aligned} \tag{190}$$

and so forth.

The first identity for X_{+1} is given by:

$$\begin{aligned}
& q^j v_1 \left(x_{-1}\right)^{-1} \ \langle \widehat{D}_- \, \phi_1 \left(q^{1/2} v_1, q^{-1/2} x_{-1}, q^{-1/2} x_{+1}, q^{1/2} \overline{v}_1\right) \\
& \times \ \phi_2 \left(q^{1/2} v_2, q^{-1/2} x_{-2}, q^{1/2} x_{+2}, q^{-1/2} \overline{v}_2\right) \rangle_q \\
& + \ q^m \langle \widehat{D}_{\overline{v}} \phi_1 \left(q^{1/2} v_1, q^{-1/2} x_{-1}, q^{1/2} x_{+1}, q^{-1/2} \overline{v}_1\right) x_{+1} \left(\overline{v}_2\right)^{-1} \\
& \times \ \phi_2 \left(q^{1/2} v_2, q^{-1/2} x_{-2}, q^{1/2} x_{+2}, q^{-1/2} \overline{v}_2\right) \rangle_q \\
& + \ q^j v_2 \left(x_{-2}\right)^{-1} \ \langle \phi_1 \left(q^{1/2} v_1, q^{-1/2} x_{-1}, q^{1/2} x_{+1}, q^{-1/2} \overline{v}_{-1}\right) \\
& \times \ \widehat{D}_- \phi_2 \left(q^{1/2} v_2, q^{-1/2} x_{-2}, q^{-1/2} x_{+2}, q^{1/2} \overline{v}_2\right) \rangle_q \\
& + \ q^m \langle \phi_1 \left(q^{1/2} v_1, q^{-1/2} x_{-1}, q^{1/2} x_{+1}, q^{-1/2} \overline{v}_1\right) x_{+2} \left(\overline{v}_2\right)^{-1} \\
& \times \ \widehat{D}_{\overline{v}} \phi_2 \left(q^{1/2} v_2, q^{-1/2} x_{-2}, q^{1/2} x_{+2}, q^{-1/2} \overline{v}_1\right) \rangle_q = 0
\end{aligned} \tag{191}$$

and five other q-difference equations.

The solution of these q-difference equations exists when the conformal dimensions d_1 and d_2 are equal: $d_1 = d_2 = d$ and is determined uniquely up to a constant. Let us use the twistors $Y = Y^\mu \sigma_\mu$. More explicitly, the matrices Y are given by:

$$Y = \begin{pmatrix} x_0 + x_3 & x_0 - i x_2 \\ x_0 + i x_2 & x_0 - x_3 \end{pmatrix} = \begin{pmatrix} x_+ & v \\ \overline{v} & x_- \end{pmatrix}, \tag{192}$$

where $x_+, x_-, v, \overline{v}$ are q-deformed Minkowski coordinates defined in Equ. (176).

It is easy to see that the quantum determinant[17]

$$det_q Y = x_- x_+ - q^{-1} v \overline{v} \quad \text{and} \quad det_q \left(Y_1 - Y_2\right) = det_q \, Y_1 \left(I - Y_1^{-1} Y_2\right), \tag{193}$$

where I is a 2×2 identity matrix. The q-deformed two-point conformal correlation function reads

$$\langle \phi_1 \left(Y_1\right) \phi_2 \left(Y_2\right) \rangle_q = C\left(q\right) \ \left(det_q \, Y_1\right)^{-d} \quad {}_1\varphi_0 \left(d; q^{1-d/2} Y_1^{-1} Y_2\right), \tag{194}$$

where $C\left(q\right)$ is a constant and where the quantum hypergeometric function

[17] We use Manin's notation [26].

with matricial argument[18] is given by:

$$ {}_1\varphi_0\left(d;Y\right)=det_q\prod_{l=0}^{\infty}\left(I-q^lY\right)^{-1}\left(I-q^{d+l}Y\right). \tag{195} $$

One easily sees that the q-correlation function reduces to the undeformed conformal correlation function because ${}_1\varphi_0\left(d;Y\right)$ becomes ${}_1F_0\left(d;Y\right)=det\left(I-Y\right)^{-d}$ in the limit $q\to 1$.

The identities for the q-deformed three-point conformal correlation functions read

$$ \begin{aligned} &\left(\pi\left(X_i\right)\otimes q^{\pi(-H_i/2)}\otimes q^{\pi(-H_i/2)}+q^{\pi(-H_i/2)}\otimes\pi\left(X_i\right)\otimes q^{\pi(-H_i/2)}\right.\\ &\left.+q^{\pi(-H_i/2)}\otimes q^{\pi(-H_i/2)}\otimes\pi\left(X_i\right)\right)\left\langle\phi_1\left(Y_1\right)\phi_2\left(Y_2\right)\phi_3\left(Y_3\right)\right\rangle_q=0, \end{aligned} \tag{196} $$

$$ \begin{aligned} &\left(\pi\left(\mathcal{D}\right)\otimes\pi\left(\mathcal{D}\right)\otimes\pi\left(\mathcal{D}\right)\right)\left\langle\phi_1\left(Y_1\right)\phi_2\left(Y_2\right)\phi_3\left(Y_3\right)\right\rangle_q=\\ &\left\langle\phi_1\left(Y_1\right)\phi_2\left(Y_2\right)\phi_3\left(Y_3\right)\right\rangle_q. \end{aligned} \tag{197} $$

The solutions are given by

$$ \begin{aligned} &\left\langle\phi_1\left(Y_1\right)\phi_2\left(Y_2\right)\phi_3\left(Y_3\right)\right\rangle_q=C_{ijk}\\ &\left(det_qY_1\right)^{-\gamma_{12}^3}\ {}_1\varphi_0\left(\gamma_{12}^3;q^{1-d_1/2}Y_1^{-1}Y_2\right).\\ &\left(det_qY_2\right)^{-\gamma_{23}^1}\ {}_1\varphi_0\left(\gamma_{23}^1;q^{1-d_2/2}Y_2^{-1}Y_3\right).\\ &\left(det_qY_1\right)^{-\gamma_{31}^2}\ {}_1\varphi_0\left(\gamma_{31}^2;q^{1+\frac{d_2-d_1}{2}}Y_1^{-1}Y_3\right), \end{aligned} \tag{198} $$

where $\gamma_{ij}^k=\frac{d_k-d_i-d_j}{2}$ and C_{ijk} are the structure constants.

Let us mention once more that at $q=1$ we get the ordinary conformal correlation functions.
These results are the first steps towards the construction of the quantum bulk to boundary and bulk to bulk propagators in the quantum $\mathrm{AdS}_5^q/\mathrm{CFT}_4^q$ correspondence.

[18]classical hypergeometric functions with matricial argument were introduced by Bochner [59] through Bessel functions with matricial argument. They have been used in generating probability distributions as a generalization of the use of classical hypergeometric functions [60].

10 Conclusion and Outlook

Quantum Groups emerged as generalized symmetries in the study of quantum integrable systems. Ever since, they become the cornerstone in theoretical physics. Recent discoveries of Vafa and his collaborators [37] clearly prove that q-deformation will play an important role in string theory, Yang-Mills theory and Black holes. The aim of this review was to present a recent and pedagogical overview of the definitions and methods used in quantum groups and quantum gauge theory and highlight some recent results found by the author. On going and envisaged work involves investigations of the q-deformed string theory, not just by deforming the oscillators, but by using the formal properties of the Hopf algebra structures. Next, we will try to deeply understand the connection of these new q-strings to Vafa's work.

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